Trigonometrik fonksiyonların integrallerinden başka fonksiyonların integralleri için de indirgeme formülleri elde edilebilir.

örnek: $\int \frac{dx}{(1+x^2)^n}$ tipindeki integral için indirgeme formülü bulalım:

$$I_{n-1} = \int \frac{dx}{(1+x^2)^{n-1}} \quad , \quad \left(\begin{array}{l} u = \frac{1}{(1+x^2)^{n-1}} \Rightarrow du = \frac{(1-n).2x}{(1+x^2)^n}dx \\ dv = dx \Rightarrow v = x \end{array} \right)$$ kısmi int. uygulanırsa

$$I_{n-1} = \int \frac{dx}{(1+x^2)^{n-1}} = \frac{x}{(1+x^2)^{n-1}} - 2.(1-n).\int \frac{x^{2+1-1}dx}{(1+x^2)^n}$$

$$= \frac{x}{(1+x^2)^{n-1}} - 2.(1-n).\int \frac{dx}{(1+x^2)^{n-1}} + 2.(1-n).\int \frac{dx}{(1+x^2)^n}$$

$$= \frac{x}{(1+x^2)^{n-1}} + 2.(n-1).I_{n-1} - 2.(n-1).I_n$$

$$I_n = \frac{1}{2n-2}.\frac{x}{(1+x^2)^{n-1}} + \frac{2n-3}{2n-2}I_{n-1}$$ yani

$$\boxed{\int \frac{dx}{(1+x^2)^n} = \frac{1}{2n-2}.\frac{x}{(1+x^2)^{n-1}} + \frac{2n-3}{2n-2}.\int \frac{dx}{(1+x^2)^{n-1}}}$$

olarak bulunur.

örnek: $$\int \frac{dx}{(1+x^2)^3} = \frac{1}{4}.\frac{x}{(1+x^2)^2} + \frac{3}{4}.\int \frac{dx}{(1+x^2)^2}$$

$$= \frac{1}{4}.\frac{x}{(1+x^2)^2} + \frac{3}{4}\left[\frac{1}{2}.\frac{x}{1+x^2} + \frac{1}{2}\int \frac{dx}{1+x^2}\right]$$

$$= \frac{1}{4}\frac{x}{(1+x^2)^2} + \frac{3}{8}.\frac{x}{1+x^2} + \frac{3}{8}.\arctan x + C$$

Benzer şekilde hareket ederek bu tip integrallerin daha genel hali için indirgeme formülü:

$$\boxed{\int \frac{dx}{(a^2+x^2)^n} = \frac{1}{2n-2}.\frac{x}{a^2(a^2+x^2)^{n-1}} + \frac{2n-3}{2n-2}.\frac{1}{a^2}.\int \frac{dx}{(a^2+x^2)^{n-1}}} \quad (5.1.)$$

olarak bulunur.

örnek: $$\int \frac{dx}{(9+x^2)^3} = \frac{1}{4}.\frac{1}{9}\frac{x}{(3^2+x^2)^2} + \frac{3}{4}.\frac{1}{9}\int \frac{dx}{(9+x^2)^2}$$

$$= \frac{x}{36.(9+x^2)^2} + \frac{1}{12}\int \frac{dx}{(9+x^2)^2} = \frac{x}{36(9+x^2)^2} + \frac{1}{12}\left[\frac{1}{2}.\frac{1}{9}.\frac{x}{(9+x^2)} + \frac{1}{2}\frac{1}{9}\int \frac{dx}{9+x^2}\right.$$

$$= \frac{x}{36(9+x^2)^2} + \frac{1}{216(9+x^2)} + \frac{1}{216}.\frac{1}{3}.\arctan\frac{x}{3} + C$$

$I=\int \frac{dx}{(ax^2+bx+c)^n}$, $b^2-4ac<0$ tipindeki integraller için indirgeme formülü elde edelim:

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x \overset{-(\frac{b}{2a})^2+(\frac{b}{2a})^2}{+} \frac{c}{a}\right] = a\cdot\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

$$\left(\begin{array}{l} u=x+\frac{b}{a} \\ t^2=\frac{4ac-b^2}{4a^2}\end{array}\right) \quad I=\int\frac{dx}{(ax^2+bx+c)^n}=\frac{1}{a^n}\int\frac{du}{(t^2+u^2)^n}$$

(5.1 formülü kullanılarak int. bulunur.)

Ödev problemleri:

1) $\int \sin^{3/2}x \cdot \cos x\, dx$ 2) $\int \sec^4 3x \cdot \tan 3x\, dx$ 3) $\int \sqrt{1-\sin x}\, dx$

4) $\int \csc\frac{x}{2}\cdot\cot\frac{x}{2}\, dx$ 5) $\int \frac{\sin^3 x}{\cos^2 x}\, dx$ 6) $\int (1+\cos 3x)^{3/2} dx$

5.6. Trigonometrik değişken değiştirme

Üç temel trigonometrik değişken değiştirme sözkonusudur.

a^2-u^2 tipindeki ifadeler için $u=a\sin t$ (veya $u=a\tanh t$)

a^2+u^2 " " " $u=a\tan t$ (veya $u=a\sinh t$)

u^2-a^2 " " " $u=a\sec t$ (veya $u=a\cosh t$)

değişken dönüşümleri yapılır.

a^2-u^2 cebirsel ifadesini bulunduran integrallerin hesabı:

$u=a\sin t \Rightarrow du=a\cos t\,dt$, $\sqrt{a^2-u^2}=\sqrt{a^2-a^2\cos^2 t}=|a|\cdot\sqrt{1-\sin^2 t}=|a||\cos t|$

$(t\in(-\pi/2,\pi/2) \Rightarrow \cos t>0 \Rightarrow \sqrt{a^2-u^2}=|a|\cdot\cos t$ olur.

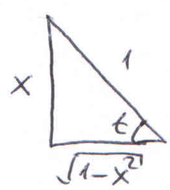
$\boxed{\int\frac{du}{\sqrt{a^2-u^2}}=\arcsin\frac{u}{a}+c}$ olduğunu hatırlatalım.

Örnek: 1) $\int\frac{x^3}{\sqrt{1-x^2}}dx$, $x=\sin t \Rightarrow dx=\cos t\,dt$

$$\int\frac{x^3}{\sqrt{1-x^2}}dx=\int\frac{\sin^3 t\cdot\cos t\,dt}{\sqrt{1-\sin^2 t}}=\int\sin^3 t\,dt=\int\sin t\cdot(1-\cos^2 t)\,dt$$

$$=-\int(1-v^2)\,dv=-v+\frac{v^3}{3}+c=-\cos t+\frac{1}{3}\cos^3 t+c$$

$$=-\sqrt{1-x^2}+\frac{1}{3}(1-x^2)^{3/2}+c$$



$\cos t=\sqrt{1-x^2}$

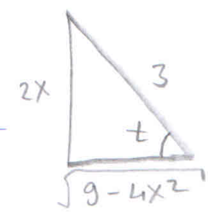
2) $\int \frac{1}{x^2 \cdot \sqrt{9-x^2}}\,dx$, $x = 3\sin t$, $dx = 3\cos t\,dt$

$$\int \frac{dx}{x^2 \cdot \sqrt{9-x^2}} = \int \frac{3\cos t\,dt}{9\sin^2 t \cdot \sqrt{9-9\sin^2 t}} = \frac{1}{9}\int \frac{\cos t}{\sin^2 t \cdot \cos t}\,dt = \frac{1}{9}\int \frac{dt}{\sin^2 t}$$

$$= \frac{1}{9}\int \operatorname{cosec}^2 t\,dt = -\frac{1}{9}\cot t + c = -\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + c$$

3) $\int \sqrt{9-4x^2}\,dx$, $2x = 3\sin t \Rightarrow 2dx = 3\cos t\,dt$

$$\int \sqrt{9-4x^2}\,dx = \frac{3}{2}\int \sqrt{9-9\sin^2 t} \cdot \cos t\,dt = \frac{9}{2}\int \cos^2 t\,dt$$



$$= \frac{9}{2} \cdot \frac{1}{2}\int (1+\cos 2t)\,dt = \frac{9}{4}\left(t + \frac{1}{2}\sin 2t\right) + c$$

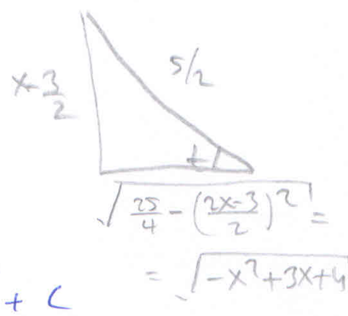
$$= \frac{9}{4}t + \frac{9}{8}\sin 2t + c = \frac{9}{4}\arcsin\frac{2x}{3} + \frac{9}{8} \cdot 2 \cdot \sin t \cdot \cos t + c$$

$$= \frac{9}{4}\arcsin\frac{2x}{3} + \frac{9}{4} \cdot \frac{2x}{3} \cdot \frac{\sqrt{9-4x^2}}{3} + c$$

$$= \frac{9}{4}\arcsin\frac{2x}{3} + \frac{1}{2} \cdot x\sqrt{9-4x^2} + c$$

4) $\int \sqrt{-x^2+3x+4}\,dx = ?$ $\left(-x^2+3x+4 = \left(\frac{5}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2 \quad x - \frac{3}{2} = \frac{5}{2}\sin t \Rightarrow dx = \frac{5}{2}\cos t\,dt \right)$

$$\int \sqrt{-x^2+3x+4}\,dx = \int \sqrt{\left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 \sin^2 t} \cdot \frac{5}{2}\cos t\,dt$$



$$= \frac{25}{4}\int \cos^2 t\,dt = \frac{25}{4} \cdot \frac{1}{2}\int (1+\cos 2t)\,dt$$

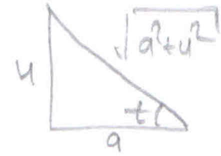
$$= \frac{25}{8}t + \frac{25}{8} \cdot \frac{1}{2}\sin 2t + c$$

$$= \frac{25}{8}\arcsin\left(\frac{2x-3}{5}\right) + \frac{2x-3}{8}\sqrt{-x^2+3x+4} + c$$

a^2+u^2 cebirsel ifadesini bulunduran integrallerin hesaplanması:

Bu durumda $a>0$ olmak üzere $u=a\tan t$ dönüşümü yapılır. $du=a\sec^2 t\,dt$, $1+\tan^2 t=\sec^2 t$ özdeşliği kullanılır.

örnek: 1) $a>0$, $\int \frac{du}{\sqrt{a^2+u^2}}$, $u=a\tan t \Rightarrow du=a\sec^2 t\,dt$



$$\int \frac{du}{\sqrt{a^2+u^2}} = \int \frac{a\sec^2 t\,dt}{\sqrt{a^2\sec^2 t}} = \int \sec t\,dt = \ln|\sec t+\tan t| + c$$

$$= \ln\left|\frac{\sqrt{a^2+u^2}}{a} + \frac{u}{a}\right| + c = \ln\left|\sqrt{a^2+u^2}+u\right| + C, \quad \boxed{C=c-\ln a}$$

2) $\int \frac{dx}{(4x^2+9)^2} = ?$ $\left(2x=3\tan t,\ x=\frac{3}{2}\tan t,\ dx=\frac{3}{2}\sec^2 t\,dt\right)$

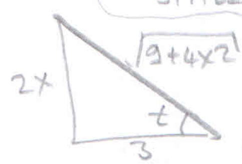
$$\int \frac{dx}{(4x^2+9)^2} = \frac{3}{2}\int \frac{\sec^2 t\,dt}{(9\tan^2 t+9)^2} = \frac{1}{54}\int \frac{dt}{\sec^2 t} = \frac{1}{54}\int \cos^2 t\,dt =$$

$$= \frac{1}{54}\cdot\frac{1}{2}\int(1+\cos 2t)\,dt = \frac{1}{54\cdot 2}t + \frac{1}{54}\cdot\frac{1}{2}\cdot\frac{1}{2}\sin 2t + c$$

$$= \frac{1}{108}\arctan\frac{2x}{3} + \frac{1}{108}\cdot\cos t\cdot\sin t + c$$

$$= \frac{1}{108}\arctan\frac{2x}{3} + \frac{x}{18\cdot(4x^2+9)} + c$$

$\sin t = \frac{2x}{\sqrt{9+4x^2}}$



$\cos t = \frac{3}{\sqrt{9+4x^2}}$

3) $\int \sqrt{x^2+4x+5}\,dx = ?$ $\left(x^2+4x+5=(x+2)^2+1,\ x+2=\tan t,\ dx=\sec^2 t\,dt\right)$

$$\int \sqrt{x^2+4x+5}\,dx = \int \sqrt{(x+2)^2+1}\,dx = \int \sqrt{\tan^2 t+1}\,\sec^2 t\,dt$$

$$= \int \sec^3 t\,dt = \frac{1}{2}\sec t\tan t + \frac{1}{2}\ln|\sec t+\tan t| + c$$

$$= \frac{x+2}{2}\sqrt{x^2+4x+5} + \frac{1}{2}\ln\left|\sqrt{x^2+4x+5}+x+2\right| + c$$

4) $\int \frac{dx}{\sqrt{x^2-2x+5}} = ?$ $x^2-2x+5=(x-1)^2+4$, $x-1=2\tan t$, $dx=2\sec^2 t\,dt$

$$\int \frac{dx}{\sqrt{x^2-2x+5}} = \int \frac{dx}{\sqrt{(x-1)^2+2^2}} = \int \frac{2\sec^2 t\,dt}{\sqrt{4\tan^2 t+4}} = \int \frac{\sec^2 t\,dt}{\sec t} = \int \sec t\,dt$$

$$= \ln|\sec t+\tan t| + c = \ln\left|\frac{\sqrt{x^2-2x+5}}{2} + \frac{x-1}{2}\right| + c$$

5) $\int \frac{dx}{\sqrt{9x^2-12x+20}} = ?$ $\left(3x-2=4\tan t\quad dx=\frac{4}{3}\sec^2 t\,dt\right)$

$$\int \frac{dx}{\sqrt{9x^2-12x+20}} = \frac{4}{3}\int \frac{\sec^2 t\,dt}{\sqrt{16\tan^2 t+16}} = \frac{1}{3}\int \sec t\,dt = \frac{1}{3}\ln|\sec t+\tan t| + c$$

$$= \frac{1}{3}\cdot\ln\left|\frac{\sqrt{9x^2-12x+20}}{4} + \frac{3x-2}{4}\right| + c$$

u^2-a^2 ifadesini bulunduran integrallerin hesaplanması:

Bu durumda $u=a\sec t \Rightarrow du=a.\sec t.\tan t\,dt$, $\sec^2 t-1=\tan^2 t$ kullanılır.

$$\int \frac{du}{\sqrt{u^2-a^2}} = \int \frac{a\sec t\tan t}{\sqrt{a^2\tan^2 t}}\,dt = \pm\int \sec t\,dt = \pm \ln|\sec t+\tan t|+c$$

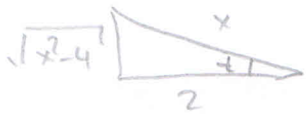
$$=\pm \ln\left|\frac{u}{a}\pm\frac{\sqrt{u^2-a^2}}{a}\right|+c = \begin{cases}\ln\left|\frac{u}{a}+\frac{\sqrt{u^2-a^2}}{a}\right|+c\\ -\ln\left|\frac{u}{a}-\frac{\sqrt{u^2-a^2}}{a}\right|+c\end{cases}$$

(Gerçekte bu iki ifade birbirine eşittir.)

Örnek: 1) $\int \frac{dx}{x^2.\sqrt{x^2-4}} = ?$ $\quad x=2\sec t,\ dx=2\sec t.\tan t\,dt$

$$\int \frac{dx}{x^2.\sqrt{x^2-4}} = \int \frac{2\sec t.\tan t\,dt}{4\sec^2 t.\sqrt{4\sec^2 t-4}} = \frac{1}{4}\int \frac{\tan t\,dt}{\sec t.\sqrt{\tan^2 t}} = \frac{1}{4}\int \frac{dt}{\sec t}$$

$$=\frac{1}{4}\int \cos t\,dt = \frac{1}{4}\sin t+c = \frac{\sqrt{x^2-4}}{4x}+c$$



2) $\int \sqrt{4x^2-4x-3}\,dx = ?$ $\quad 4x^2-4x-3=(2x-1)^2-4$, $2x-1=2\sec t \Rightarrow$

$$\int\sqrt{4x^2-4x-3}\,dx = \int\sqrt{(2x-1)^2-4}\,dx = \int\sqrt{4\sec^2 t-4}\,.\sec t.\tan t\,dt$$

$$=2.\int \sec t.\tan^2 t\,dt = 2.\int \sec t.(\sec^2 t-1)\,dt =$$

$$=2.\int \sec^3 t\,dt - 2.\int \sec t\,dt =$$

$$\int\sqrt{4x^2-4x-3}\,dx = 2.\left\{\frac{1}{2}\sec t\tan t+\frac{1}{2}\ln|\sec t+\tan t|\right\} - 2.\ln|\sec t+\tan t|+c$$

$$=\sec t.\tan t - \ln|\sec t+\tan t|+c$$

$$=\frac{(2x-1).\sqrt{4x^2-4x-3}}{4} - \ln\left|\frac{2x-1}{2}+\frac{\sqrt{4x^2-4x-3}}{2}\right|+c$$

3) $\int \frac{dx}{\sqrt{x^2-4x-5}} = ?$ $\quad x-2=3\sec t \Rightarrow dx=3\sec t\tan t\,dt$

$$\int \frac{dx}{\sqrt{x^2-4x-5}} = \int \frac{dx}{\sqrt{(x-2)^2+9}} = \int \frac{3\sec t\tan t\,dt}{\sqrt{9\sec^2 t-9}} = \int \sec t\,dt$$

$$=\ln|\sec t+\tan t|+c = \ln\left|\frac{x-2}{3}+\frac{\sqrt{x^2-4x-5}}{3}\right|+c$$

4) $\int \frac{x+3}{\sqrt{2x^2-8x}}\,dx = ?$ $\quad u=x-2 \quad du=dx$

$$\int \frac{x+3}{\sqrt{2x^2-8x}}\,dx = \int \frac{x+3}{\sqrt{2.(x-2)^2-8}}\,dx = \int \frac{u+5}{\sqrt{2u^2-8}}\,du = \frac{1}{\sqrt{2}}\int\frac{u+5}{\sqrt{u^2-4}}\,du =$$

$$=\frac{1}{\sqrt{2}}\int\frac{u}{\sqrt{u^2-4}}\,du + \frac{5}{\sqrt{2}}\int\frac{du}{\sqrt{u^2-4}}$$

(birinci integralde $v=u^2-4$, $dv=\cancel{\ }\,2u\,du$, ikinci integralde $u=2\sec t$

$$\int \frac{x+3}{\sqrt{2x^2-8x}} = \frac{1}{2\sqrt{2}} \int \frac{dv}{\sqrt{v}} + \frac{5}{\sqrt{2}} \cdot \int \frac{2\sec t \tan t\, dt}{\sqrt{4\sec^2 t - 4}}$$

$$= \frac{1}{\sqrt{2}} \cdot \sqrt{v} + \frac{5}{\sqrt{2}} \cdot \int \sec t\, dt$$

$$= \frac{1}{\sqrt{2}} \cdot \sqrt{u^2-4} + \frac{5}{\sqrt{2}} \cdot \ln|\sec t + \tan t| + c$$

$$= \frac{1}{\sqrt{2}} \sqrt{(x-2)^2-4} + \frac{5}{\sqrt{2}} \ln\left| \frac{u}{2} + \frac{\sqrt{u^2-4}}{2} \right| + c$$

$$= \frac{1}{\sqrt{2}} \sqrt{(x-2)^2-4} + \frac{5}{\sqrt{2}} \ln\left| \frac{x-2+\sqrt{(x-2)^2+4}}{2} \right| + c$$

$$= \frac{1}{2} \sqrt{2x^2-8x} + \frac{5}{\sqrt{2}} \cdot \ln\left| \frac{x-2+\sqrt{x^2-4x}}{2} \right| + c$$

<u>ax^2+bx+c $(a \neq 0)$ ifadesini bulunduran integraller!</u>

ax^2+bx+c ifadesi uygun işlemlerle $a.(u^2+B)$ ifadesine dönüştürülür. Şöyle ki;

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x\right)+c = a\left[x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right]+c-\frac{b^2}{4a^2}$$

$$= a\left[x+\frac{b}{2a}\right]^2 + \frac{4ac-b^2}{4a}$$

$u = x+\frac{b}{2a}$, $B = \frac{4ac-b^2}{4a} \Rightarrow ax^2+bx+c = a(u^2+B)$ olur.

. Eğer ax^2+bx+c karekök içinde ise bu ifade negatif olmamalıdır.

Eğer a negatif B pozitif ise karekök tanımlı olmayacağından iki hal söz konusudur.

<u>1. hali:</u> a pozitif ise $\sqrt{a(u^2+B)} = \sqrt{a}\cdot\sqrt{u^2+B}$ olur ki bu tip integralin nasıl alınacağı anlatıldı.

<u>2. hali:</u> a ve B negatif ise $-a$ ve $-B$ pozitif olacağından

$\sqrt{a(u^2+B)} = \sqrt{-a(-u^2-B)} = \sqrt{-a}\cdot\sqrt{-B-u^2}$, $-B^2 = A$ denirse

$= +\sqrt{-a}\cdot\sqrt{A^2-u^2}$ olur ki bu integralin nasıl alınacağı biliniyor.

Örnekler: 1) $\int \frac{dx}{\sqrt{2x-x^2}} = ?$ $\left(\begin{aligned} 2x-x^2 &= -(x^2-2x) = -[x^2-2x+1-1] \\ &= -[(x-1)^2-1] = 1-(x-1)^2 \end{aligned} \right)$

$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + c = \arcsin(x-1)+c$$

$u = x-1$
$du = dx$

2) $\int \frac{x+1}{\sqrt{2x^2-6x+4}} = ?$

$$2x^2-6x+4 = 2(x^2-3x+2) = 2\left[x^2-3x+2+\frac{9}{4}-\frac{9}{4}\right)$$
$$= 2.\left(x^2-3x+\left(\tfrac{3}{2}\right)^2\right)+4-\frac{9}{2}$$
$$= 2\left[(x-\tfrac{3}{2})^2-\frac{1}{4}\right]$$

$u = x-\frac{3}{2}$, $du = dx$

$$\int \frac{(x+1)dx}{\sqrt{2x^2-6x+4}} = \frac{1}{\sqrt{2}}\int \frac{x+1}{\sqrt{(x-\frac{3}{2})^2-\frac{1}{4}}} = \frac{1}{\sqrt{2}}\int \frac{u+5/2}{\sqrt{u^2-(\frac{1}{2})^2}}\,du$$

$$= \frac{1}{\sqrt{2}}\cdot\int \frac{u\,du}{\sqrt{u^2-(\frac{1}{2})^2}} + \frac{5}{2\sqrt{2}}\int \frac{du}{\sqrt{u^2-(\frac{1}{2})^2}}$$

$t = u^2-\frac{1}{4}$
$dt = 2u\,du$
$u\,du = \frac{1}{2}dt$

$$= \frac{1}{2\sqrt{2}}\int \frac{dt}{\sqrt{t}} + \frac{5}{2\sqrt{2}}\ln\left|\frac{u}{a}+\frac{\sqrt{u^2-a^2}}{a}\right| + c_2$$

$$= \frac{1}{2\sqrt{2}}\sqrt{t}+c_1+\ \cdots$$

$$= \sqrt{\frac{u^2-\frac{1}{4}}{2}} + \frac{5}{2\sqrt{2}}\ln\left|\frac{x-3/2}{1/2}+\frac{\sqrt{(x-3/2)^2-\frac{1}{4}}}{1/2}\right| + C$$

Ödev problemler: Aşa. int. hesaplayınız.

1) $\int \frac{dx}{8+2x^2}$ 2) $\int \frac{dx}{\sqrt{9-4x^2}}$ 3) $\int \frac{3\,dt}{\sqrt{9t^2-1}}$

4) $\int \frac{dx}{x\sqrt{x^2-3}}$ 5) $\int \frac{x\,dx}{(x^2-1)^{3/2}}$ 6) $\int \frac{dx}{x^2-2x+5}$

7) $\int \frac{dx}{\sqrt{x^2-2x+5}}$