Uygulama 12

1) $f: [2,5] \to \mathbb{R}$, $f(x) = x^2 - \frac{16}{3}x$ fonksiyonu verilsin. $[2,5]$ aralığını 3 eşit parçaya bölerek oluşturulan P bölüntüsü için $A(f,P)$ ve $\ddot{U}(f,P)$ toplamlarını bulunuz.

<u>Çözüm:</u> $n=3$

Alt aralıklar: $[2,3]$, $[3,4]$, $[4,5]$

$\Delta x_1 = \Delta x_2 = \Delta x_3 = 1$

$P = \{2, 3, 4, 5\}$

$f(x) = x^2 - \frac{16}{3}x$

$f'(x) = 2x - \frac{16}{3}$

$f'(x) = 0 \iff x = \frac{8}{3} \in [2,3]$

x		2		$\frac{8}{3}$		3		4		5	
$f'(x)$	$-$		$-$	0	$+$		$+$		$+$		$+$
			Azalan		Artan						

$x = \frac{8}{3}$ yerel min. nokta

$A(f,P) = m_1 \cdot \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 = \left(-\frac{64}{9} - 7 - \frac{16}{3}\right) \cdot 1 = \frac{-175}{9}$

$m_1 = \inf\{f(x) : x \in [2,3]\} = f\left(\frac{8}{3}\right) = -\frac{64}{9}$

$m_2 = \inf\{f(x) : x \in [3,4]\} = f(3) = -7$

$m_3 = \inf\{f(x) : x \in [4,5]\} = f(4) = -\frac{16}{3}$

$\ddot{U}(f,P) = M_1 \cdot \Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3 = \left(\frac{-20}{3} - \frac{16}{3} - 5\right) \cdot 1 = -\frac{41}{3}$

$M_1 = \sup\{f(x) : x \in [2,3]\} = \max\{f(2), f(3)\} = \max\left\{-\frac{20}{3}, -7\right\} = -\frac{20}{3}$

$M_2 = \sup\{f(x) : x \in [3,4]\} = f(4) = \frac{-16}{3}$

$M_3 = \sup\{f(x) : x \in [4,5]\} = f(5) = \frac{-5}{3}$

2) $f:[0,2\pi]\to\mathbb{R}$, $f(x)=\sin x+\cos x$ fonksiyonu verilsin. $[0,2\pi]$ aralığını 4 eşit parçaya bölerek oluşturulan P bölüntüsü için $\ddot{U}(f,P)$ toplamını bulunuz.

Çözüm: $n=4$

Alt aralıklar: $\left[0,\frac{\pi}{2}\right]$, $\left[\frac{\pi}{2},\pi\right]$, $\left[\pi,\frac{3\pi}{2}\right]$, $\left[\frac{3\pi}{2},2\pi\right]$

$\Delta x_1=\Delta x_2=\Delta x_3=\Delta x_4=\frac{\pi}{2}$

$P=\left\{0,\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi\right\}$

$f(x)=\sin x+\cos x \Rightarrow f'(x)=\cos x-\sin x$

$f'(x)=0 \Leftrightarrow \sin x=\cos x$, $x\in[0,2\pi]$

$\Leftrightarrow x=\frac{\pi}{4}$, $x=\frac{5\pi}{4}$

x	0		$\frac{\pi}{4}$		$\frac{\pi}{2}$		π		$\frac{5\pi}{4}$		$\frac{3\pi}{2}$		2π	
$f'(x)$	+	+	0	−		−		−	0	+		+		+
		$\nearrow$		$\searrow$		$\searrow$		$\searrow$		$\nearrow$		$\nearrow$		
		Artan		Azalan						Artan				

$x=\frac{\pi}{4}$ yerel max. nokta

$x=\frac{5\pi}{4}$ yerel min. nokta

3) $\displaystyle\int_0^3 ([|x|]+5)^{[|x|]}\,dx = ?$

Çözüm: $\displaystyle\int_0^3 ([|x|]+5)^{[|x|]}\,dx = \int_0^1 (0+5)^0\,dx + \int_1^2 (1+5)^1\,dx + \int_2^3 (2+5)^2\,dx$

$\displaystyle = \int_0^1 1\,dx + \int_1^2 6\,dx + \int_2^3 49\,dx$

$= 1+6+49$

$=56$

4) $\displaystyle\int_{-\pi}^{\pi} \underbrace{|\cos^3 x|}_{\text{çift fonksiyon}} dx = 2 \cdot \int_0^{\pi} |\cos^3 x|\, dx = 2\left(\int_0^{\pi/2} \cos^3 x\, dx - \int_{\pi/2}^{\pi} \cos^3 x\, dx\right)$

$= 2\left(\frac{2}{3} - \left(-\frac{2}{3}\right)\right)$

$= 2 \cdot \frac{4}{3} = \frac{8}{3}$

$\cos^3 x = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$

x	0		$\pi/2$		π
cos x		+	0	−	

$\displaystyle\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx = \int (1-\sin^2 x) \cos x\, dx$

$= \int (1-u^2)\, du$

$\sin x = u$

$\cos x\, dx = du$

$= u - \frac{u^3}{3} + c$

$= \sin x - \frac{\sin^3 x}{3} + c$

$\displaystyle\int_0^{\pi/2} \cos^3 x\, dx = \left(\sin x - \frac{\sin^3 x}{3}\right)\Big|_0^{\pi/2} = 1 - \frac{1}{3} = \frac{2}{3}$

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$\displaystyle\int_{\pi/2}^{\pi} \cos^3 x\, dx = \left(\sin x - \frac{\sin^3 x}{3}\right)\Big|_{\pi/2}^{\pi} = -\left(1 - \frac{1}{3}\right) = -\frac{2}{3}$

5) $\displaystyle\int_0^2 (|x-1| + 1)\, dx$ integralini tanımdan yararlanarak bulunuz.

Çözüm: $\displaystyle\int_0^2 (|x-1|+1)\, dx = \int_0^1 (|x-1|+1)\, dx + \int_1^2 (|x-1|+1)\, dx$

$\displaystyle = \int_0^1 (1-x+1)\, dx + \int_1^2 (x-1+1)\, dx$

$\displaystyle = \int_0^1 (2-x)\, dx + \int_1^2 x\, dx = \frac{3}{2} + \frac{3}{2} = 3$

x		1	
x-1	−	0	+

$\displaystyle\int_0^1 (2-x)\, dx = ?$

$P_1 = \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n}\}$, $\|P_1\| \to 0$

$f(x) = 2-x$

$f'(x) = -1 < 0 \Rightarrow f$ azalan

$[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \ldots, [\frac{n-1}{n}, \frac{n}{n}]$

$m_1 = \inf\{f(x) : x \in [0, \frac{1}{n}]\} = f(\frac{1}{n}) = 2 - \frac{1}{n}$

$m_2 = \inf\{f(x) : x \in [\frac{1}{n}, \frac{2}{n}]\} = f(\frac{2}{n}) = 2 - \frac{2}{n}$

$\vdots$

$m_n = \inf\{f(x) : x \in [\frac{n-1}{n}, \frac{n}{n}]\} = f(\frac{n}{n}) = 2 - \frac{n}{n}$

$M_1 = \sup\{ f(x) : x \in [0, \frac{1}{n}] \} = f(0) = 2-0$

$M_2 = \sup\{ f(x) : x \in [\frac{1}{n}, \frac{2}{n}] \} = f(\frac{1}{n}) = 2 - \frac{1}{n}$

$\vdots$

$M_n = \sup\{ f(x) : x \in [\frac{n-1}{n}, \frac{n}{n}] \} = f(\frac{n-1}{n}) = 2 - \frac{n-1}{n}$

$$A(f, P_1) = \left(2 - \frac{1}{n} + 2 - \frac{2}{n} + \dots + 2 - \frac{n}{n}\right) \frac{1}{n}$$

$$= \left(2n - \frac{1 + \dots + n}{n}\right) \frac{1}{n}$$

$$= \left(2n - \frac{n \cdot (n+1)}{2\cancel{n}}\right) \cdot \frac{1}{n} = 2 - \frac{n+1}{2n}$$

$$Ü(f, P_1) = \left(2 - 0 + 2 - \frac{1}{n} + \dots + 2 - \frac{n-1}{n}\right) \cdot \frac{1}{n}$$

$$= \left(2n - \frac{1 + \dots + (n-1)}{n}\right) \cdot \frac{1}{n}$$

$$= \left(2n - \frac{(n-1) \cdot \cancel{n}}{2\cancel{n}}\right) \cdot \frac{1}{n} = 2 - \frac{n-1}{2n}$$

$$\lim_{n\to\infty} A(f, P_1) = \lim_{n\to\infty} Ü(f, P_1) = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\int_1^2 x\,dx = ?$$

$P_2 = \{ 1, 1 + \frac{1}{n}, \dots, 1 + \frac{n}{n} = 2 \}$, $\|P_2\| \to 0$

$[1, 1+\frac{1}{n}]$, $[1 + \frac{1}{n}, 1 + \frac{2}{n}]$, ..., $[1 + \frac{n-1}{n}, 1 + \frac{n}{n}]$

$g(x) = x \Rightarrow g'(x) = 1 > 0 \Rightarrow g$ artan

$m_1 = g(1) = 1$

$m_2 = g(1 + \frac{1}{n}) = 1 + \frac{1}{n}$

$\vdots$

$m_n = g(1 + \frac{n-1}{n}) = 1 + \frac{n-1}{n}$

$M_1 = g(1 + \frac{1}{n}) = 1 + \frac{1}{n}$

$M_2 = g(1 + \frac{2}{n}) = 1 + \frac{2}{n}$

$\vdots$

$M_n = g(1 + \frac{n}{n}) = 1 + \frac{n}{n}$

$$A(g, P_2) = \left(1 + 1 + \frac{1}{n} + \dots + 1 + \frac{n-1}{n}\right) \cdot \frac{1}{n} = \left(n + \frac{(n-1) \cdot \cancel{n}}{2\cancel{n}}\right) \cdot \frac{1}{n} = 1 + \frac{n-1}{2n}$$

$$Ü(g, P_2) = \left(1 + \frac{1}{n} + \dots + 1 + \frac{n}{n}\right) \frac{1}{n} = \left(n + \frac{n \cdot (n+1)}{2\cancel{n}}\right) \cdot \frac{1}{n} = 1 + \frac{n+1}{2n}$$

$$\lim_{n\to\infty} A(g, P_2) = \lim_{n\to\infty} Ü(g, P_2) = 1 + \frac{1}{2} = \frac{3}{2}$$

6) $f(x)=\begin{cases} 1, & x\in Q \\ 0, & x\notin Q \end{cases}$ fonksiyonu $[0,1]$ de integrallenebilir mi? Tanımdan yararlanarak gösteriniz.

Çözüm: $[0,1]$ aralığını n eşit parçaya bölelim.

$$P=\left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}=1\right\}$$

$$\left[0,\frac{1}{n}\right], \left[\frac{1}{n},\frac{2}{n}\right], \left[\frac{2}{n},\frac{3}{n}\right], \dots, \left[\frac{n-1}{n},\frac{n}{n}\right]$$ Alt aralıklar

$$\|P\| = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} \to 0, \quad \Delta x_1 = \Delta x_2 = \dots = \Delta x_n = \frac{1}{n}$$

$m_1 = \inf\{f(x) : x\in [0,\frac{1}{n}]\} = 0$ $\quad M_1 = \sup\{f(x) : x\in [0,\frac{1}{n}]\} = 1$

$m_2 = \inf\{f(x) : x\in [\frac{1}{n},\frac{2}{n}]\} = 0$ $\quad M_2 = \sup\{f(x) : x\in [\frac{1}{n},\frac{2}{n}]\} = 1$

$\vdots \qquad\qquad \vdots$

$m_n = \inf\{f(x) : x\in [\frac{n-1}{n},\frac{n}{n}]\} = 0$ $\quad M_n = \sup\{f(x) : x\in [\frac{n-1}{n},\frac{n}{n}]\} = 1$

$A(f,P) = (0+\dots+0)\frac{1}{n} = 0, \quad Ü(f,P) = (1+\dots+1)\frac{1}{n} = 1$

$\lim\limits_{n\to\infty} A(f,P) = 0 \neq 1 = \lim\limits_{n\to\infty} Ü(f,P) \Rightarrow f, [0,1]$ de integrallenebilir değil.