Seventh Edition

VECTOR MECHANICS FOR ENGINEERS:
STATICS

CHAPTER

9

10. HAFTA

Ferdinand P. Beer, E. Russell Johnston, Jr. &
R.C. HIBBELER ' in STATICs kitaplarından
düzenlenmiştir.

Düzenleyen:
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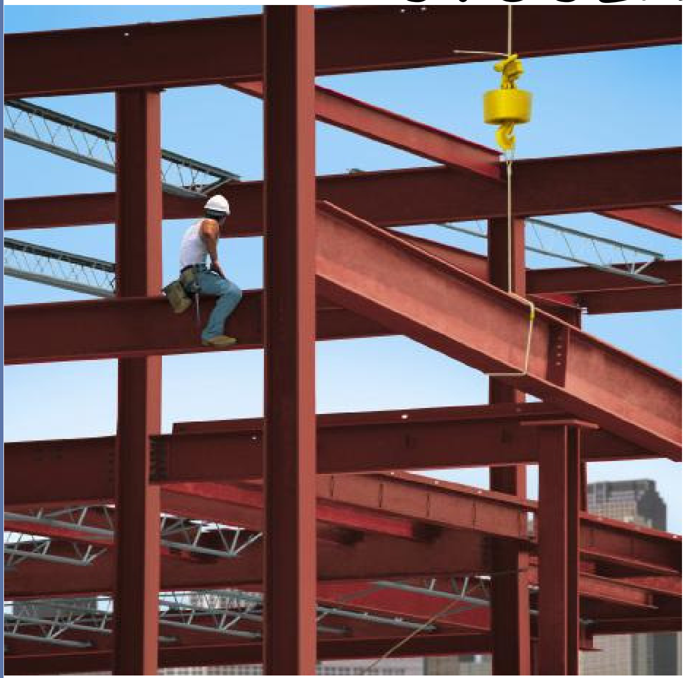
Atalet Momentleri
(Moments of Inertia)

GİRİŞ

Yapı elemanlarının mukavemeti büyük oranda onların kesit büyüklüklerine ve şekline bağlıdır, özellikle de kesit alanının 2. momenti veya atalet momentine bağlıdır.

UYGULAMALAR



Kiriş ve kolon gibi bir çok yapı elemanı have cross sectional shapes like I, H, C, etc. Diğerleri içi dolu kare yada daire kesitlerden çok tüb şeklindedir.

Why do they usually not have solid rectangular, square, or circular cross sectional areas?

- Tasarımda bu elemanların hangi özellikleri daha etkendir ?
- How can we calculate this property?

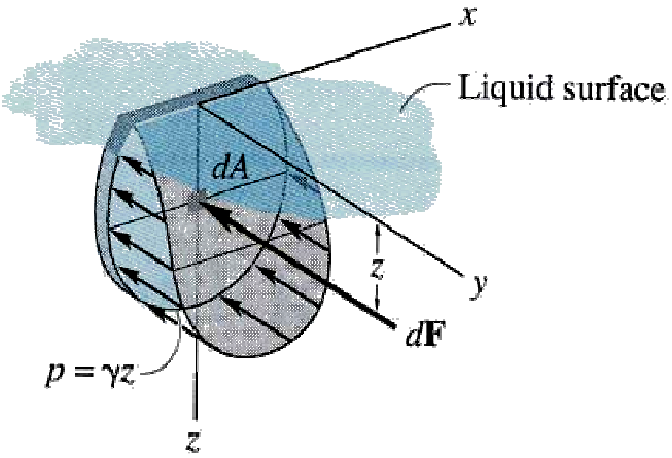
Atalet Momenti, Statik hesaplarında kullanılmaz ama hesaplanması ağırlık merkezi hesaplarına benzediği için statik dersi içinde gösterilir. Atalet momenti daha çok mukavemet (malzeme mekaniği), dinamik ve akışkanlar mekaniği derslerinde ve hesaplamalarında kullanılır.

• Bir cismin ***atalet momenti*** onun dönmeye karşı direncinin bir ölçümüdür. Günlük tecrübelerimizden de biliriz ki dönen büyük bir tekerleği durdurmak veya dönmeye başlatmak küçük tekerlekten daha zordur. Matematiksel olarak da bu olayın büyük tekerleğin daha büyük atalet momentine sahip olması nedeniyle olduğu gösterilebilir.

•Atalet momenti çeşitli mühendislik hesaplamalarında kullanılır;

- *Hidrostatik basınç kuvvetlerinin bileşkesinin yerini bulmak için,*
- *Kirişlerde gerilme ve sehim hesapları için,*
- *Dönen cisimlerin kütle atalet momentleri hesabı*

MOMENTS OF INERTIA FOR AREAS



Sıvı içine daldırılmış bir plağı göz önüne alalım. Yüzeyden z kadar aşağıdaki sıvı basıncı $p=\gamma z$ ile verilir. where γ is the specific weight of the liquid.

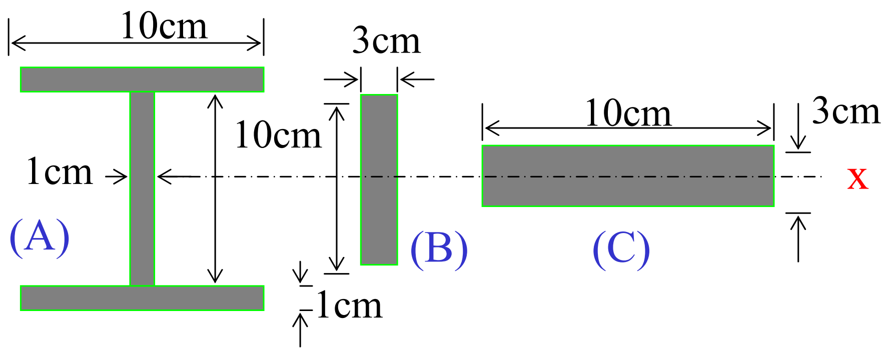
Bu noktada dA alanına etki eden kuvvet $dF = p\,dA = (\gamma\, z)\, dA$.

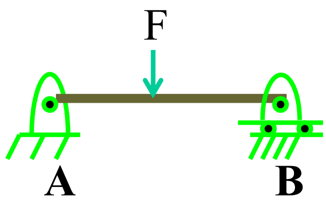
Bu kuvvet nedeniyle x-eksenine göre moment z(dF) dir.

The total moment is $\int_A z\,dF = \int_A \gamma\, z^2\,dA = \gamma \boxed{\int_A (z^2\,dA)}$.

This integral term is referred to as the moment of inertia of the area of the plate about an axis.

MOMENTS OF INERTIA FOR AREAS





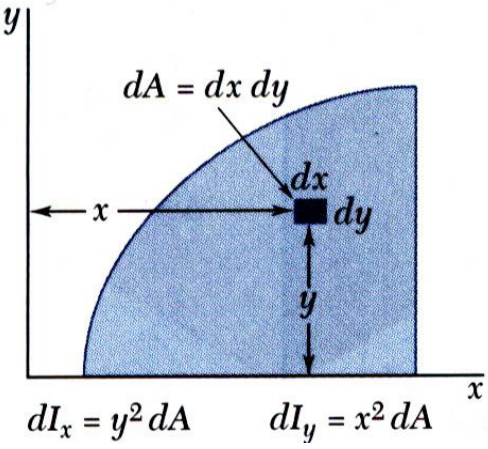
AB kirişi için 3 farklı kesit şekli ve alanı göz önüne alalım. Toplam alanlar eşit ve aynı malzemeden yapılmış, dolayısıyla birim uzunluk için kütleleri aynı.

- Verilen yükleme hali için hangisini tercih edersiniz ? Niçin ?

(daha az gerilme ve çökmeyi dikkate alın).

Yanıt x-eksenine göre atalet momentine bağlıdır. x-ekseninden en uzak alanların çoğu (A) da olduğu için en büyük atalet momentine sahip olan (A) dır. Bu nedenle de en az gerilme ve çökmeyi (δ) veren de A şıkkıdır. Atalet momenti arttıkça (δ) ve gerilme düşer.

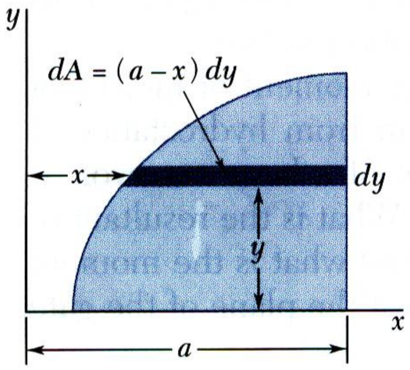
İntegrasyonla bir alanın atalet momentinin bulunması :



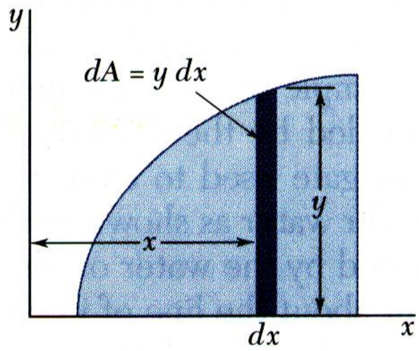
- Herhangi bir alanın x ve y eksenlerine göre *ikinci Momenti* or *atalet momenti*,

$$\boxed{I_x = \int y^2\, dA \qquad I_y = \int x^2\, dA}$$

- dA alanının koordinat eksenlerine parelel ince şerit halinde seçilmesi (I_x için yatay, I_y için düşey eleman) integral hesabını basitleştirir.



$dI_x = y^2 dA$

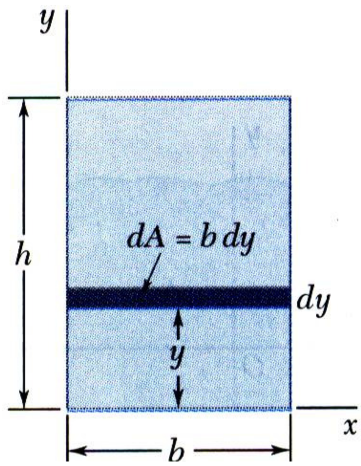


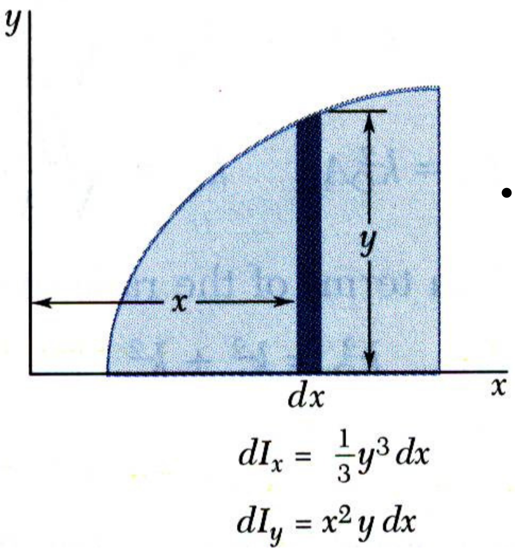
$dI_y = x^2 dA$

Moment of Inertia of an Area by Integration

- For a rectangular area,

$$I_x = \int y^2 dA = \int_0^h y^2 b\,dy = \tfrac{1}{3} b h^3$$

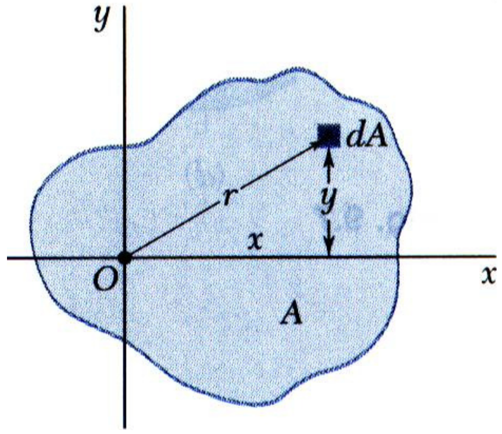




- Dikdörtgen alan için bulunan ifade eksenlere parelel olarak seçilecek ince şeritlere de uygulanabilir, örneğin bir tek düşey şerit eleman seçilerek her iki eksene göre atalet momentleri

$$\boxed{dI_x = \tfrac{1}{3} y^3 dx \qquad dI_y = x^2 dA = x^2 y\, dx}$$

Polar Moment of Inertia



- *Polar Atalet momenti,* dönen silindirik millerin burulmasında önemli bir parametredir.

$$\boxed{J_0 = \int r^2 dA}$$

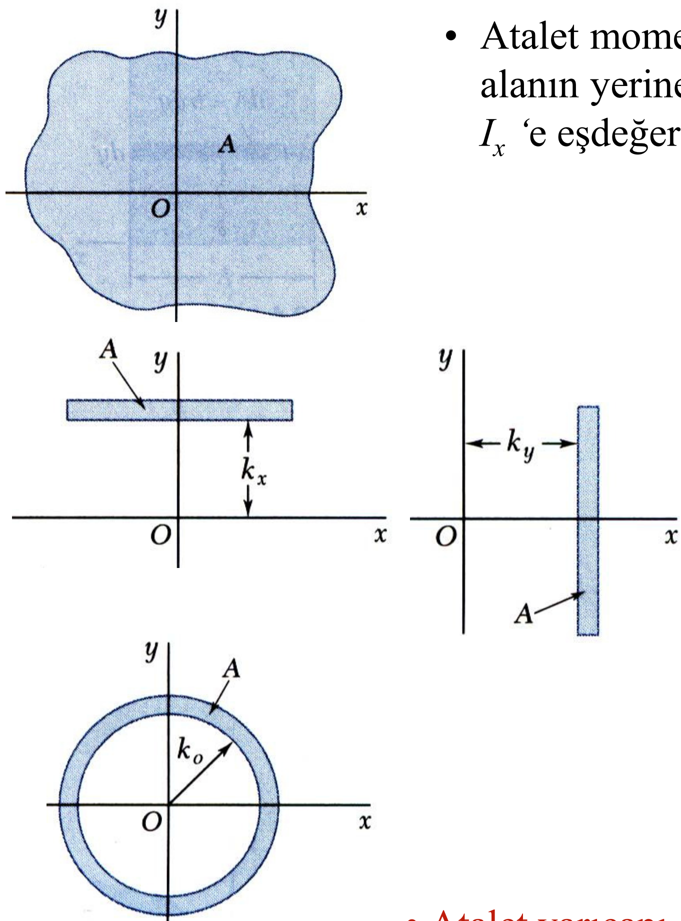
- Polar atalet momenti ile dik atalet momentleri arasındaki ilişki,

$$J_0 = \int r^2 dA = \int \left(x^2 + y^2\right) dA = \int x^2 dA + \int y^2 dA$$

$$\boxed{J_0 = I_y + I_x}$$

- Atalet momentinin birimi uzunluğun 4. kuvvetidir (m^4).

Bir alanın atalet yarıçapı



- Atalet momeni I_x olan bir alanı göz önüne alalım. Bu alanın yerine x-eksenine parelel ve atalet momenti yine I_x 'e eşdeğer bir dikdörtgen şerit düşünürsek,

$$\boxed{I_x = k_x^2 A} \qquad k_x = \sqrt{\frac{I_x}{A}}$$

$k_x =$ x eksenine göre atalet yarıçapı

- Similarly,

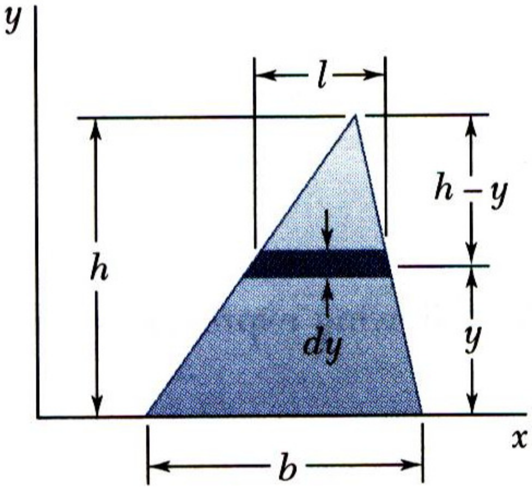
$$\boxed{I_y = k_y^2 A} \qquad k_y = \sqrt{\frac{I_y}{A}}$$

$$\boxed{J_O = k_O^2 A} \qquad k_O = \sqrt{\frac{J_O}{A}}$$

$$\boxed{k_O^2 = k_x^2 + k_y^2}$$

- Atalet yarıçapı özellikle kolonların tasarımında önemlidir.

Sample Problem 9.1



Üçgen alanın tabanına göre atalet momentini hesaplayınız.

SOLUTION:

- A differential strip parallel to the x axis is chosen for dA.

$$dI_x = y^2 dA \qquad dA = l\,dy$$

- For similar triangles,

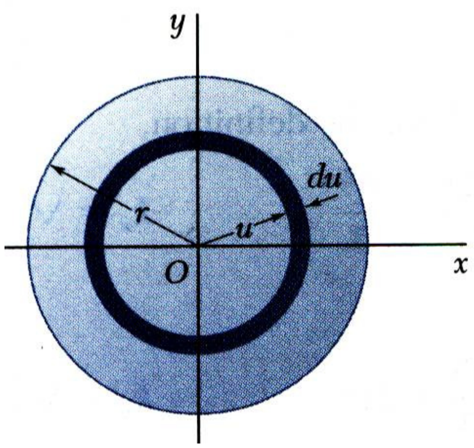
$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

- $y = 0$ dan $y = h$*' a kadar* dI_x*' in integrasyonu ,*

$$I_x = \int y^2 dA = \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h \left(hy^2 - y^3\right)dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h \qquad \boxed{I_x = \frac{bh^3}{12}}$$

Sample Problem 9.2



a) Dairesel bir alanın polar atalet momentini bulunuz.

b) *a* şıkkında bulduğunuz sonucu kullanarak, dairesel alanın çapından geçen x-eksenine göre atalet momentini bulunuz.

SOLUTION:

- An annular differential area element is chosen,

$$dJ_O = u^2 dA \qquad dA = 2\pi u\,du$$

$$J_O = \int dJ_O = \int_0^r u^2(2\pi u\,du) = 2\pi\int_0^r u^3 du$$

$$\boxed{J_O = \frac{\pi}{2}r^4}$$

- From symmetry, $I_x = I_y$,

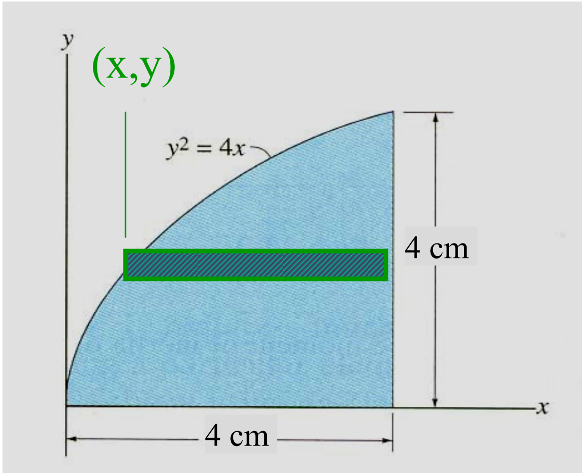
$$J_O = I_x + I_y = 2I_x \qquad \frac{\pi}{2}r^4 = 2I_x$$

$$\boxed{I_{diameter} = I_x = \frac{\pi}{4}r^4}$$

Sample Problem 9.3

Given: The shaded area shown in the figure.

Find: The MoI of the area about the x- and y-axes.



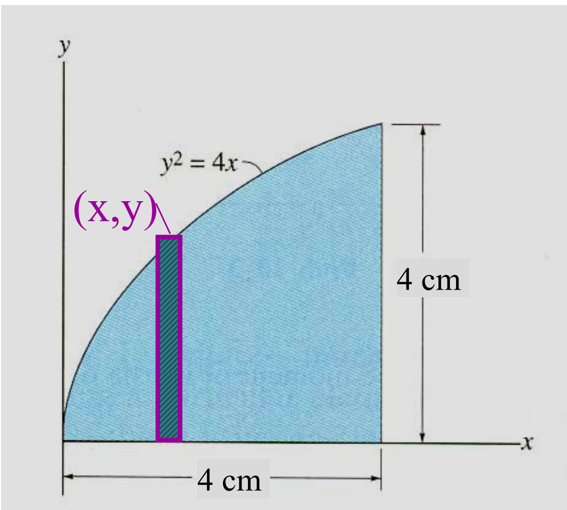
Solution

$$I_x = \int y^2 \, dA$$

$$dA = (4 - x)\,dy = (4 - y^2/4)\,dy$$

$$I_x = \int_0^4 y^2 \,(4 - y^2/4)\,dy$$

$$= \left[(4/3)\,y^3 - (1/20)\,y^5 \right]_0^4 = 34.1 \text{ cm}^4$$



$$I_y = \int x^2 \, dA = \int x^2 \; y.dx$$

$$= \int x^2 \,(2\sqrt{x})\,dx$$

$$= 2 \int_0^4 x^{2.5} \, dx$$

$$= \left[(2/3.5) \;\; x^{3.5} \right]_0^4$$

$$= 73.1 \text{ cm}^4$$

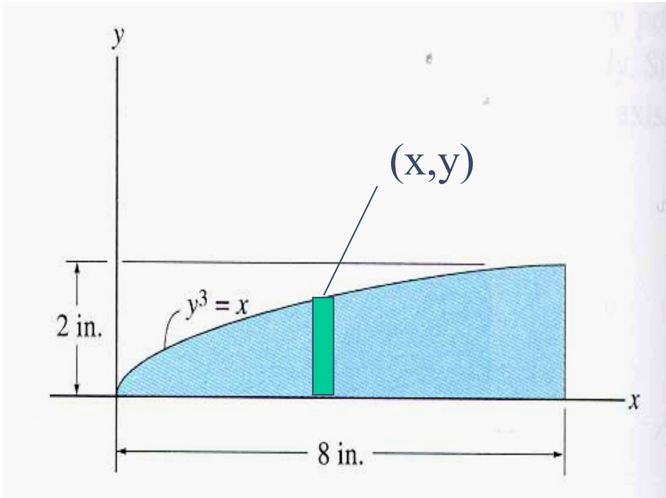
In the above example, it will be difficult to determine I_y using a horizontal strip. However, I_x in this example can be determined using a vertical strip. So,

$I_x = \int (1/3)\, y^3\, dx = \int (1/3)\,(2\sqrt{x})^3\, dx$.

Sample Problem 9.4

Given: The shaded area shown.

Find: I_x and I_y of the area.

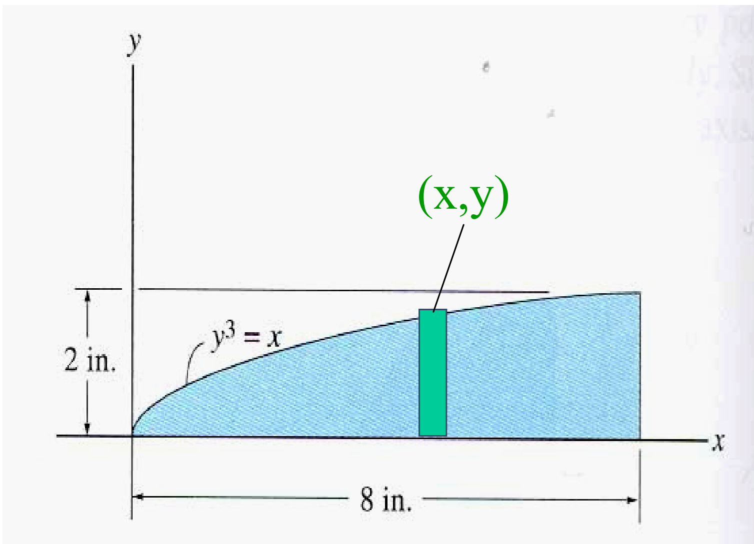


Solution

$$I_x = \int (1/3)\, y^3 \; dx$$

$$= \int_0^8 (1/3)\, x \; dx \quad = [x^2 / 6\,]_0^8$$

$$= 10.7 \quad in^4$$



$$I_Y = \int x^2 \; dA \quad = \int x^2 \; y \; dx$$

$$= \int x^2 \, (x^{(1/3)}) \; dx$$

$$= \int_0^8 x^{(7/3)} \; dx$$

$$= [(3/10) \; x^{(10/3)} \,]_0^8$$

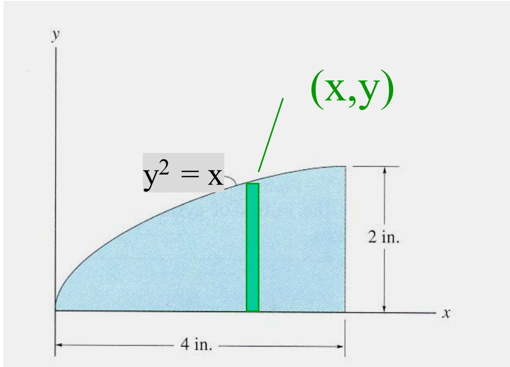
$$= 307.18 \quad in^4$$

ATTENTION QUIZ

1. When determining the MoI of the element in the figure, dI_y equals

A) x^2 dy   B) x^2 dx

C) (1/3) y^3 dx   D) $x^{2.5}$ dx

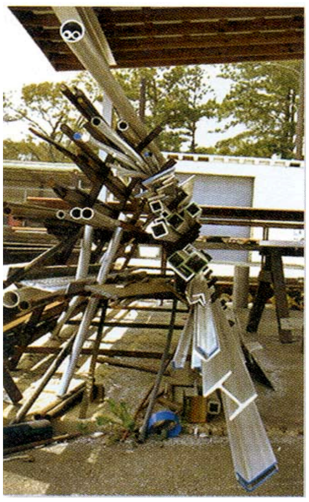


2. Similarly, dI_x equals

A) (1/3) $x^{1.5}$ dx   B) y^2 dA

C) (1/12) x^3 dy   D) (1/3) x^3 dx

Parelel eksenler Teoremi & Bileşik alanların atalet momentleri

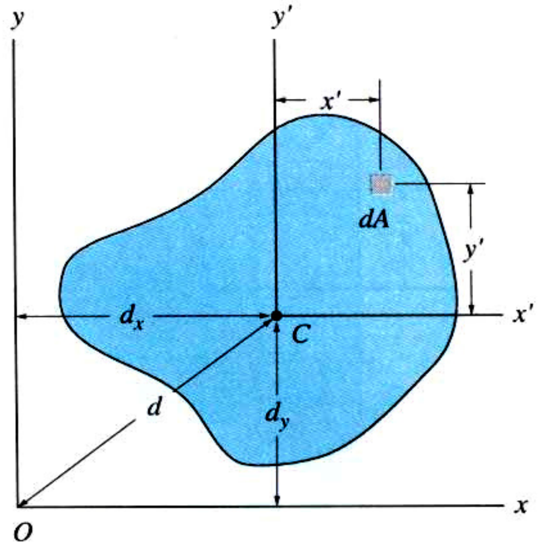


Yapı elemanlarının kesit alanları genellikle basit şekillerden yada ilkel basit şekillerin birleşmesinden oluşmuştur.

Bu basit alanların atalet momentlerini bulmak için integrasyon yöntemi ile karşılaştırıldığında daha basit yöntemler var mıdır ? ???

EVET

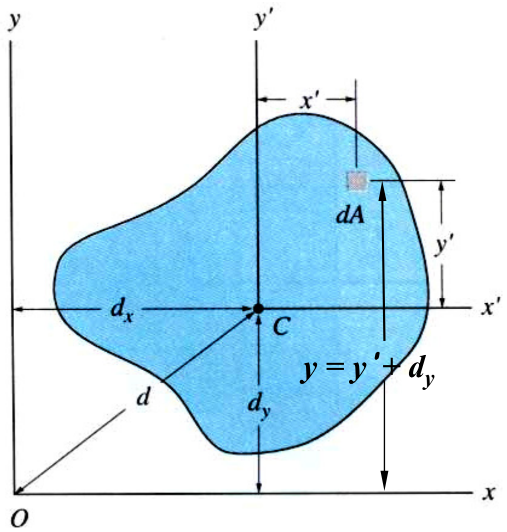
PARALLEL-AXIS THEOREM FOR AN AREA



Bu teorem bir alanın ağırlık merkezinden geçen eksenlere göre atalet momentinin yine bu eksenlere parelel başka eksenlere göre atalet momentleri ile ilgilidir. Bileşik alanların atalet momentlerinin bulunması için pratik bir yöntemdir.

Gözönüne alınan alanın ağırlık merkezi C dir. x' and y' axes ağ. mer. C' den geçmektedir. x'-eksenine parelel ve d_y kadar mesafede bir x-eksenine göre atalet momenti parelel eksen teoremi ile bulunur.

Parelel Eksen Teoremi



$$I_X = \int_A y^2\, dA = \int_A (y' + d_y)^2\, dA$$

$$= \int_A y'^2\, dA + 2\, d_y \int_A y'\, dA + d_y^2 \int_A dA$$

Ağırlık merkezinin tanımını kullanarak:

$\bar{y}' = (\int_A y'\, dA) / (\int_A dA)$. Now

since C is at the origin of the x' – y' axes,

$\bar{y}' = 0$, and hence $\int_A y'\, dA = 0$.

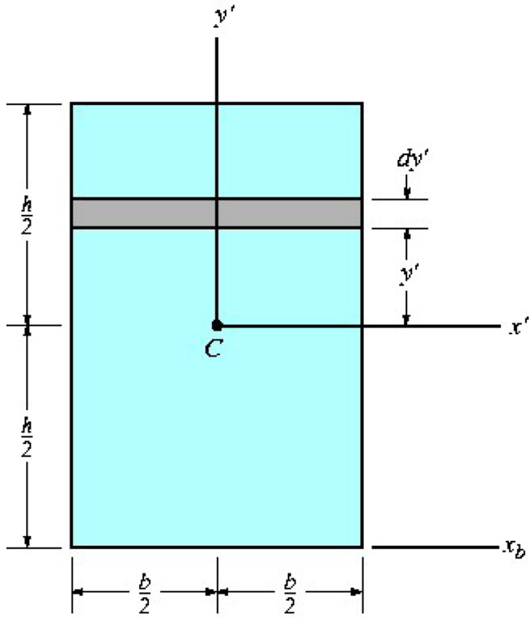
Thus $\boxed{I_X = \bar{I}_X' + A\, d_y^2}$

Similarly, $\boxed{I_Y = \bar{I}_Y' + A\, d_X^2}$ and

$\boxed{J_O = \bar{J}_C + A\, d^2}$

Example 1 *Determine the moment of inertia for the rectangular area with respect to* (a). *the centroidal x' axis,* (b). *the axis x_b passing through the base of the rectangular, and* (c). *the pole or z' axis perpendicular to the x'-y' plane and passing through the centroid C.*

Solution:



Part (a)

- *Differential element chosen, distance y' from x' axis*
- Since $dA = b\,dy'$

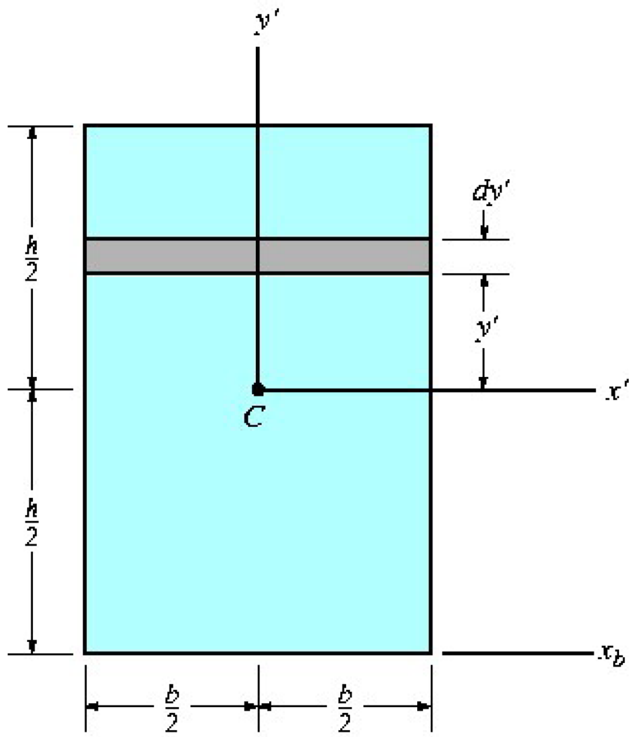
$$\bar{I}_{x'} = \int_A y'^2\,dA = \int_{-h/2}^{h/2} y'^2\,dy$$

$$= \frac{1}{12}bh^3$$

Part (b): *Moment of inertia about an axis passing through the base of the rectangle can be obtained by applying parallel axis theorem, Eq.(10-3)*

$$I_{x_b} = \bar{I}_{x'} + Ad^2$$

$$= \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3$$



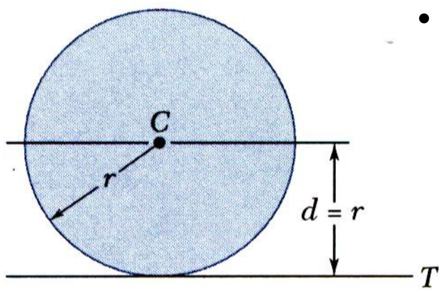
Part (c)

- *For polar moment of inertia about point C*

$$\bar{I}_{y'} = \frac{1}{12}hb^3$$

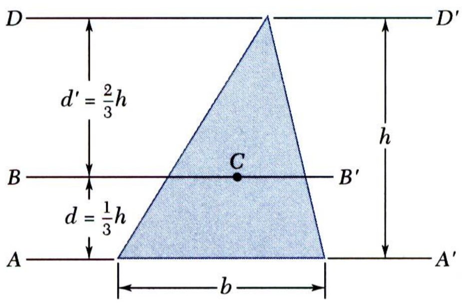
$$J_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12}bh(h^2 + b^2)$$

Parallel Axis Teoremi örnek



- Use the value of I for a circle from the table on the following page and the parallel-axis theorem to find I_T, the moment of inertia about an axis tangent to the circle.

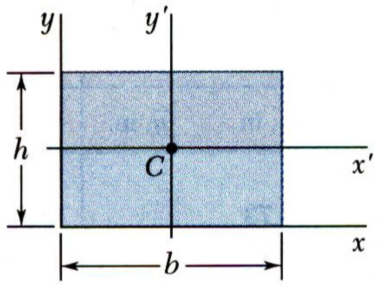
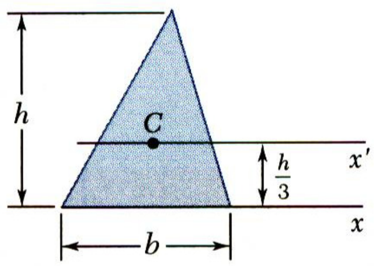
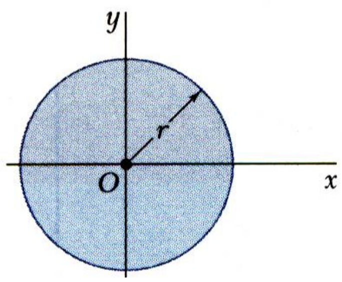
$$I_T = \bar{I} + Ad^2 = \tfrac{1}{4}\pi r^4 + \left(\pi r^2\right)r^2$$
$$= \tfrac{5}{4}\pi r^4$$



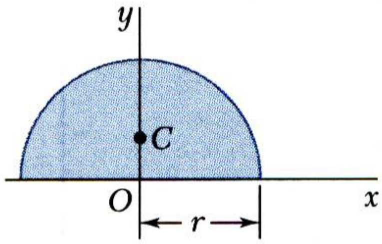
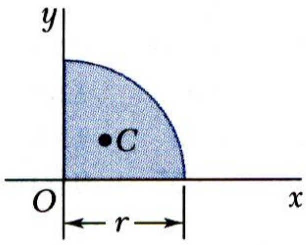
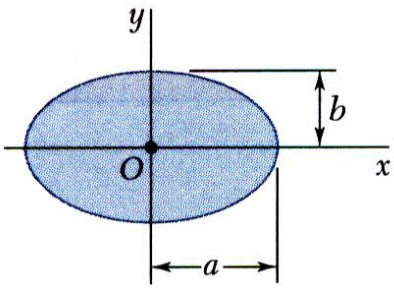
- Use the value of $I_{AA'}$ along the base of a triangle from the table on the following page and the parallel-axis theorem to find $I_{BB'}$, the moment of inertia along a parallel axis through the centroid of the triangle

$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$
$$I_{BB'} = I_{AA'} - Ad^2 = \tfrac{1}{12}bh^3 - \tfrac{1}{2}bh\left(\tfrac{1}{3}h\right)^2$$
$$= \tfrac{1}{36}bh^3$$

En çok bilinen bazı ilkel kesitlerin atalet momentleri

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$

En çok bilinen bazı ilkel kesitlerin atalet momentleri

Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi a b^3$ $\bar{I}_y = \frac{1}{4}\pi a^3 b$ $J_O = \frac{1}{4}\pi a b(a^2 + b^2)$

Example 3. *Determine the moment of inertia of the shaded area about the x axis.*

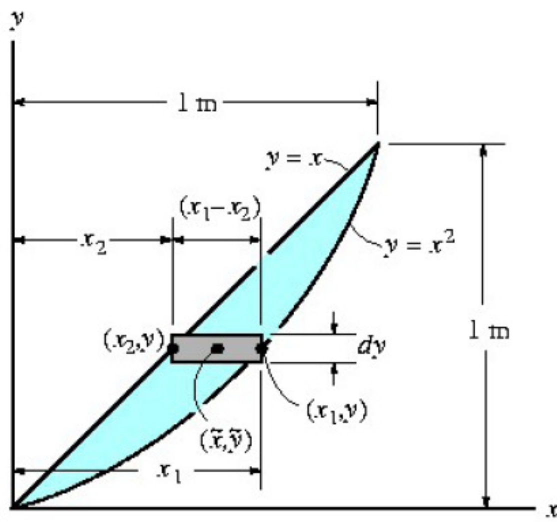
Solution:

- ✓ Differential element parallel to x axis chosen
- ✓ Intersects the curve at (x_2, y) and (x_1, y)
- ✓ Area, $dA = (x_1 - x_2)dy$
- ✓ All elements lie at the same distance y from the x axis

$$I_x = \int_A y^2 dA = \int_0^1 y^2 (x_1 - x_2) dy$$

$$= \int_0^1 y^2 \left(\sqrt{y} - y\right) dy$$

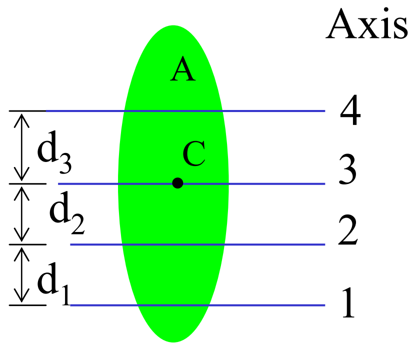
$$I_x = \frac{2}{7} y^{7/2} - \frac{1}{4} y^4 \Big|_0^1 = 0.0357 m^4$$



(a)

Fig.10-10

CONCEPT QUIZ



1. For the area A, we know the centroid's (C) location, area, distances between the four parallel axes, and the MoI about axis 1. We can determine the <u>MoI about axis 2</u> by applying the parallel axis theorem ___ .

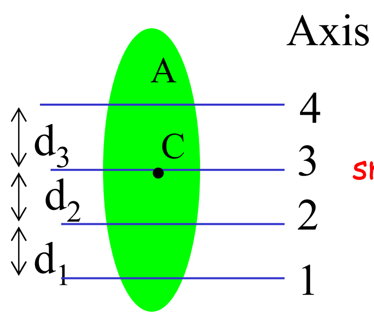
   A) directly between the axes 1 and 2.

   B) between axes 1 and 3 and then between the axes 3 and 2.

   C) between axes 1 and 4 and then axes 4 and 2.

   D) None of the above.

CONCEPT QUIZ



smallest MoI

2. For the same case, consider the MoI about each of the four axes. About which axis will the MoI be the smallest number?

   A) Axis 1

   B) Axis 2

   C) Axis 3

   D) Axis 4

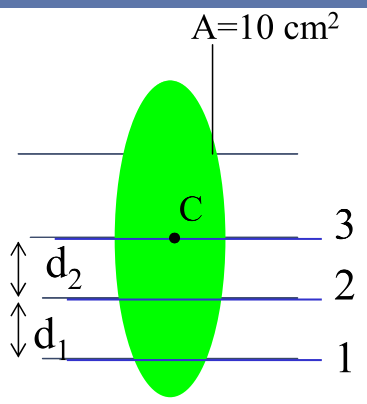
   E) Can not tell.

ATTENTION QUIZ

1. For the given area, the moment of inertia about axis 1 is 200 cm^4 . What is the MoI about axis 3 (the centroidal axis)?

   A) 90 cm^4 B) 110 cm^4

   C) 60 cm^4 D) 40 cm^4



$d_1 = d_2 = 2$ cm

$I_1 = I_3 + Ad^2$

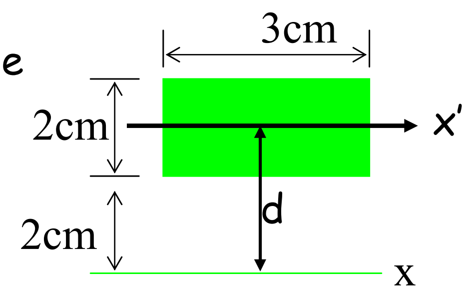
$I_3 = I_1 - Ad^2 = 200 - 10(4)^2 = 40\ cm^4$

ATTENTION QUIZ

2. The moment of inertia of the rectangle about the x-axis equals

   A) 8 cm^4 B) 56 cm^4

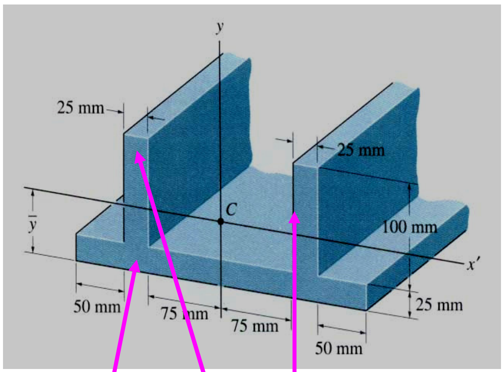
   C) 24 cm^4 D) 26 cm^4



$I_x = I_{x'} + Ad^2$

$I_x = (bh^3)/12 + Ad^2 = (3 \cdot 2^3) / 12 + 6 \cdot 3^2 = 56cm^4$

EXAMPLE



[1] [2] [3]

Given: The beam's cross-sectional area.

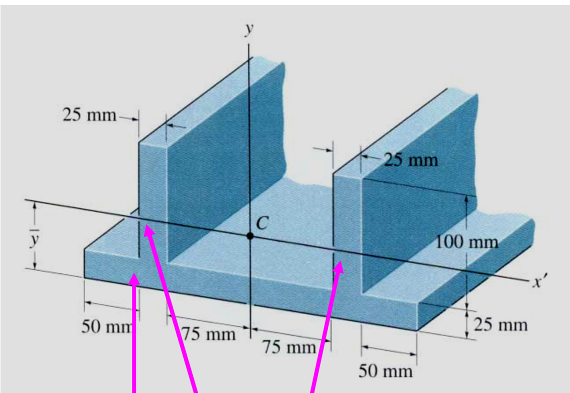
Find: The moment of inertia of the area about the y-axis and the radius of gyration k_y.

Solution

1. The cross-sectional area can be divided into three rectangles ([1], [2], [3]) as shown.

2. The centroids of these three rectangles are in their center. The distances from these centers to the y-axis are 0 mm, 87.5 mm, and 87.5 mm, respectively.

EXAMPLE



[1] [2] [3]

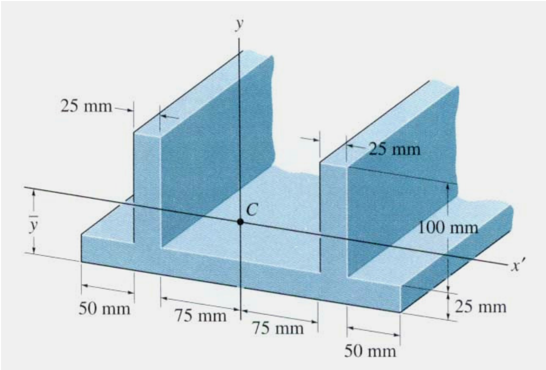
3. From the inside back cover of the book, the MoI of a rectangle about its centroidal axis is (1/12) b h^3.

$$I_{y[1]} = (1/12)\,(25\text{mm})\,(300\text{mm})^3$$
$$= 56.25\,(10^6)\ \text{mm}^4$$

Using the parallel-axis theorem,

$$I_{Y[2]} = I_{Y[3]} = \bar{I}_{Y'} + A\,(d_X)^2$$
$$= (1/12)\,(100)\,(25)^3 + (25)\,(100)\,(\,87.5\,)^2$$
$$= 19.27\ (10^6)\ \text{mm}^4$$

EXAMPLE



4. $I_y = I_{y1} + I_{y2} + I_{y3}$

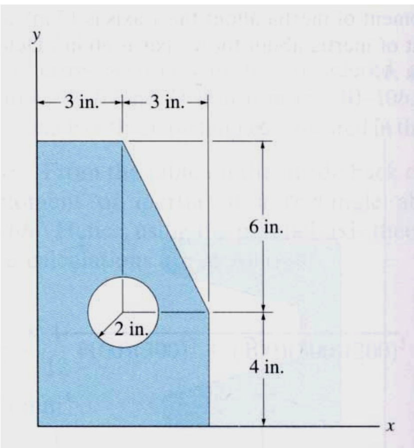
$$= 94.8\ (10^6)\ \text{mm}^4$$

$$k_y = \sqrt{(I_y / A)}$$

$$A = 300\,(25) + 25\,(100) + 25\,(100) = 12{,}500\ \text{mm}^2$$

$$k_y = \sqrt{(94.79)\ (10^6)\ /\ (12500)} = 87.1\ \text{mm}$$

MOMENT OF INERTIA FOR A COMPOSITE AREA



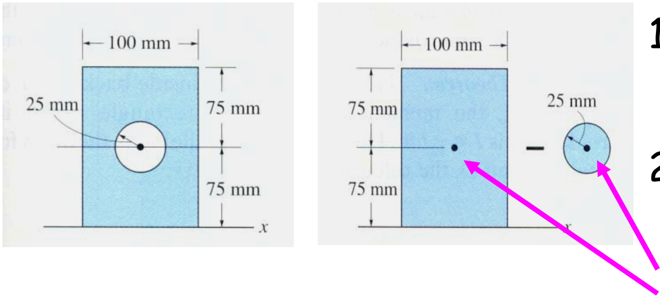
A composite area is made by adding or subtracting a series of "simple" shaped areas like rectangles, triangles, and circles.

For example, the area on the left can be made from a rectangle minus a triangle and circle.

The MoI of these "simpler" shaped areas about their centroidal axes are found in most engineering handbooks as well as the inside back cover of the textbook.

Using these data and the parallel-axis theorem, the MoI for a composite area can easily be calculated.

STEPS FOR ANALYSIS



1. Divide the given area into its simpler shaped parts.

2. Locate the centroid of each part and indicate the perpendicular distance from each centroid to the desired reference axis.

3. Determine the MoI of each "simpler" shaped part about the desired reference axis using the parallel-axis theorem ($I_X = \overline{I}_{X'} + A (d_y)^2$) .

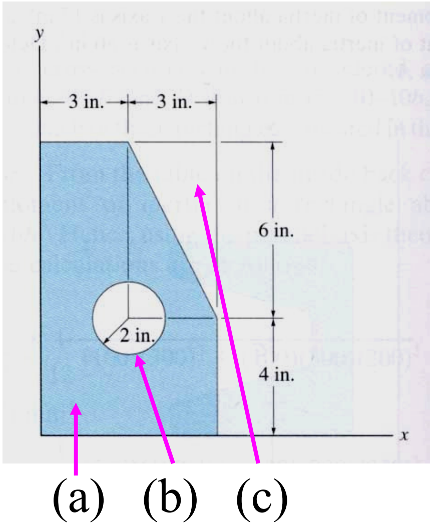
4. The MoI of the entire area about the reference axis is determined by performing an algebraic summation of the individual MoIs obtained in Step 3. (Please note that MoI of a hole is subtracted).

READING QUIZ

1. The parallel-axis theorem for an area is applied between
   - A) an axis passing through its centroid and any corresponding parallel axis.
   - B) any two parallel axis.
   - C) two horizontal axes only.
   - D) two vertical axes only.

2. The moment of inertia of a composite area equals the _____ of the MoI of all of its parts.
   - A) vector sum
   - B) algebraic sum (addition or subtraction)
   - C) addition
   - D) product

EXAMPLE



Given: The shaded area as shown in the figure.

Find: The moment of inertia for the area about the x-axis and the radius of gyration k_X.

Solution

1. The given area can be obtained by subtracting both the circle (b) and triangle (c) from the rectangle (a).
2. Information about the centroids of the simple shapes can be obtained from the inside back cover of the book. The perpendicular distances of the centroids from the x-axis are: $d_a = 5$ in , $d_b = 4$ in, and $d_c = 8$in.

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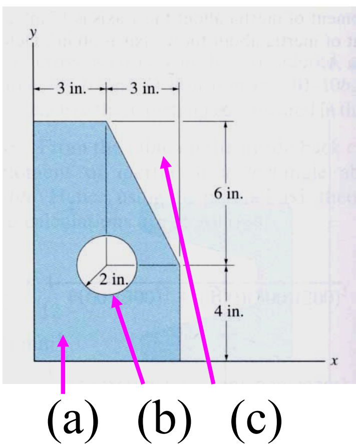
EXAMPLE



3. $I_{Xa} = (1/12)\ 6\ (10)^3 + 6\,(10)(5)^2$

$= 2000 \text{ in}^4$

$I_{Xb} = (1/4)\,\pi\,(2)^4 + \pi\,(2)^2\,(4)^2$

$= 213.6 \text{ in}^4$

$I_{Xc} = (1/36)\,(3)\,(6)^3 +$

$(½)\,(3)\,(6)\,(8)^2 = 594 \text{ in}^4$

$$I_X = I_{Xa} - I_{Xb} - I_{Xc} = 1190 \text{ in}^4$$

$$k_X = \sqrt{(I_X / A)}$$

$$A = 10\,(6) - \pi\,(2)^2 - (½)\,(3)\,(6) = 38.43 \text{ in}^2$$

$$k_X = \sqrt{(1192 / 38.43)} = 5.57 \text{ in.}$$