Rastgele Değişkenlerin Beklenen Değeri

Kesikli rastgele değişken I için “beklenti” ya da rastgele değişkenin “beklenen değer” i ağırlıklı ortalamadır.

$$E[I] = \sum_{i=-\infty}^{\infty} i\, f_I[i]$$

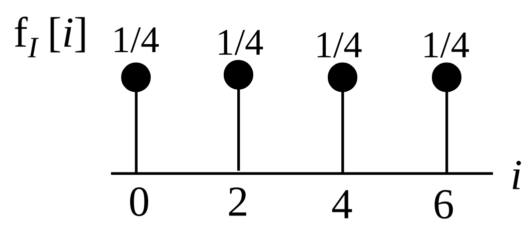
Sürekli rastgele değişken X için “beklenti” ya da rastgele değişkenin “beklenen değer” i sürekli ağırlıklı ortalamadır.

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)dx$$

<u>Örnek</u>: 0, 2, 4, 6 değerlerini alan *I* kesikli rastgele değişkenini ele alalım.

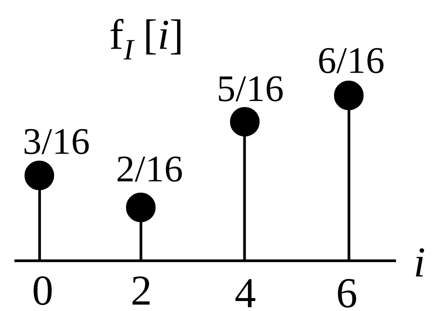
a) *I* birbiçim dağılımlı ise; beklentisi aritmetik ortalaması olur:

$$E[I] = \sum_{i=-\infty}^{\infty} i\, f_I[i] = \sum_{i=0}^{6} i\, f_I[i]$$
$$= \frac{0+2+4+6}{4} = 3$$



b) Eğer *I* nın kesikli oyf si şekildeki gibiyse, beklenen değeri

$$E[I] = \sum_{i=-\infty}^{\infty} i\, f_I[i] = \sum_{i=0}^{6} i\, f_I[i]$$
$$= 0\times\frac{3}{16} + 2\times\frac{2}{16} + 4\times\frac{5}{16} + 6\times\frac{6}{16}$$
$$= \frac{0+4+20+36}{16} = \frac{15}{4} = 3.75$$



Beklenen Değerin Değişmezliği

$Y = g(X)$ olsun

Bu durumda gösterilebilir ki:

$$E[Y] = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy = \int_{-\infty}^{\infty} g(x) \, f_X(x) \, dx = E\left[g(X)\right]$$

Yani:

$$\boxed{E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) \, f_X(x) \, dx}$$

Beklenen Değerin Bazı Önemli Özellikleri

Ortalama: m_X

1. $E[c] = c,$ c sabit
2. $E[cX] = cE[X]$
3. $E[X + c] = E[X] + c$
4. Duble beklenen değer:

$$E[E[X]] = E[m_X] = m_X = E[X]$$

5. İki rastgele değişkenin toplamının beklenen değeri:

$$E[X + Y] = E[X] + E[Y]$$

Rastgele Değişkenlerin "Momentleri"

n. moment: $E\left[X^n\right] \quad n = 1, 2, 3, \ldots$

n. *merkezi* moment: $E\left[(X - m_X)^n\right] \quad n = 1, 2, 3, \ldots$

$m_X = E\left[X\right]$ ("ortalama")

***n*. moment**

$$E\left[X^n\right]=\int_{-\infty}^{\infty} x^n\, f_X(x)dx \qquad E\left[I^n\right]=\sum_{i=-\infty}^{\infty} i^n\, \mathrm{f}_I[i]$$

n = 1 için (r.d. nin *ortalaması*)

$$m_X = E\left[X\right]=\int_{-\infty}^{\infty} x\, f_X(x)dx \qquad m_I = E\left[I\right]=\sum_{i=-\infty}^{\infty} i\, \mathrm{f}_I[i]$$

n = 2: için ikinci moment (r.d. nin ortalama "*gücü*")

$$E\left[X^2\right]=\int_{-\infty}^{\infty} x^2\, f_X(x)dx \qquad E\left[I^2\right]=\sum_{i=-\infty}^{\infty} i^2\, \mathrm{f}_I[i]$$

***n*. merkezi moment**

E $$E\left[(X-m_X)^n\right]=\int_{-\infty}^{\infty}(x-m_X)^n\, f_X(x)dx \qquad E\left[(I-m_I)^n\right]=\sum_{i=-\infty}^{\infty}(i-m_I)^n\, \mathrm{f}_I[i]$$

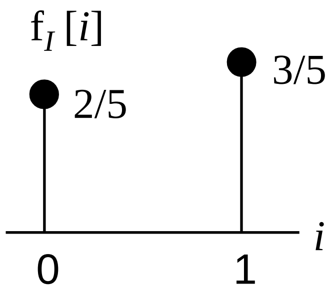
n = 2 için (r.d. nin *varyansı*)

$$\sigma_X^2=E\left[(X-m_X)^2\right]=\int_{-\infty}^{\infty}(x-m_X)^2\, f_X(x)dx$$

$$\sigma_I^2=E\left[(I-m_I)^2\right]=\sum_{i=-\infty}^{\infty}(i-m_I)^2\, f_I[i]$$

Standart sapma

$$\sigma_X=\sqrt{\sigma_X^2}$$



<u>Örnek</u>: Bernoulli r.d.

$$\Pr[1] = p = \frac{3}{5}, \qquad \Pr[0] = 1 - p = \frac{2}{5}$$

$$m_I = E[I] = \sum_{i=0}^{1} i \ \mathrm{f}_I[i] = \frac{2}{5}(0) + \frac{3}{5}(1) = \frac{3}{5}$$

$$\sigma_I^2 = E\left[\left(I - m_I\right)^2\right] = \sum_i \left(i - m_I\right)^2 \ \mathrm{f}_I[i]$$

$$= \left(0 - \frac{3}{5}\right)^2 \frac{2}{5} + \left(1 - \frac{3}{5}\right)^2 \frac{3}{5}$$

$$= \frac{9}{25} \cdot \frac{2}{5} + \frac{4}{25} \cdot \frac{3}{5} = \frac{30}{125} = \frac{6}{25}$$

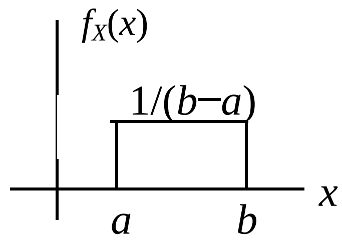
Örnek: $X\,[a, b]$ aralığında birbiçim ise m_X ve σ_X^2 nedir?

(a) $$m_X = E[X] = \frac{1}{b-a}\int_a^b x\,dx = \frac{b^2-a^2}{(b-a)2} = \frac{b+a}{2}$$

(b) $$\sigma_X^2 = E\left[\left(X-m_X\right)^2\right] = \frac{1}{b-a}\int_a^b (x-m_X)^2\,dx$$

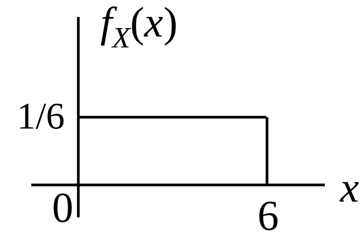
$$= \frac{1}{b-a}\cdot\frac{(x-m_X)}{3}\Bigg|_a^b = \frac{(b-m_X)^3-(a-m_X)^3}{3(b-a)}$$

$$= \frac{\left(\frac{b-a}{2}\right)^3-\left(\frac{a-b}{2}\right)^3}{3(b-a)} = \frac{\frac{1}{4}\left(b-a\right)^3}{3(b-a)} = \frac{(b-a)^2}{12}$$



$a = 0$, $b = 6$, için

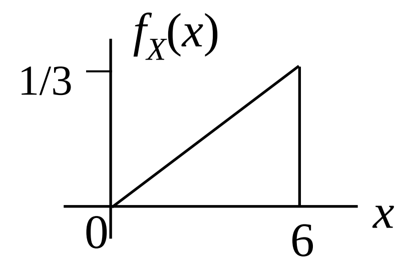
$$m_X = 3, \qquad \sigma_X^2 = \frac{(6-0)^2}{12} = 3.$$



Örnek:

Aşağıda verilen olasılık yoğunluk fonksiyonuna sahip X rastgele değişkeni için ortalama ve varyansı bulun.

$$f_X(x) = \begin{cases} \frac{1}{18}x, & 0 \le x \le 6 \\ 0, & \text{diğer} \end{cases}$$



$$m_x = E[X] = \frac{1}{18}\int_0^6 x^2 dx = \left.\frac{x^3}{54}\right|_0^6 = 4$$

$$\sigma_X^2 = E\left[\left(X - m_X\right)^2\right] = \frac{1}{18}\int_0^6 (x-4)^2 x\, dx$$

$$= \frac{1}{18}\int_0^6 (x^3 - 8x^2 + 16x)\, dx = \frac{1}{18}\left[\frac{x^4}{4} - \frac{8x^3}{3} + 8x^2\right]_0^6 = 2.$$

2.

Biraz daha Varyans: $\left(\text{var}[X] \text{ veya } \sigma_X^2\right)$

1. $\boxed{\sigma_X^2 = E\left[X^2\right] - m_X^2}$

$$
\begin{aligned}
\text{var}[X] &= E\left[\left(X - m_X\right)^2\right] \\
&= E\left[X^2 - 2m_X X + m_X^2\right] \\
&= E\left[X^2\right] - 2m_X E[X] + m_X^2 \\
&= E\left[X^2\right] - 2m_X \cdot m_X + m_X^2 \\
&= E\left[X^2\right] - m_X^2 = E\left[X^2\right] - \left\{E\left[X\right]\right\}^2
\end{aligned}
$$

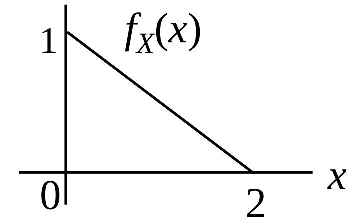
2. var $[c] = 0$

3. var $[X + c] =$ var $[X]$

4. var $[cX] = c^2$ var $[X]$

Örnek:

Ranstgele değişken X in oyf si aşağıdaki gibidir



$$f_X(x) = \begin{cases} 1 - \dfrac{x}{2}, & 0 \le X \le 2 \\ 0, & \text{diğer} \end{cases}$$

(a) X in ortalamasını, 2. Momentini ve varyansını bulun.

$$m_X = E[X] = \int_{-\infty}^{\infty} x\, f_X(x)dx = \int_0^2 x\left(1 - \frac{x}{2}\right)dx = \left[\frac{x^2}{2} - \frac{x^3}{6}\right]_0^2 = \frac{2}{3}$$

E

$$E\left[X^2\right] = \int_0^2 x^2\left(1 - \frac{x}{2}\right)dx = \left[\frac{x^3}{3} - \frac{x^4}{8}\right]_0^2 = \frac{2}{3}.$$

$$\sigma_X^2 = \text{var}[X] = \int_0^2 \left(x - \frac{2}{3}\right)^2 \left(1 - \frac{x}{2}\right)dx = \int_0^2 \left(\frac{1}{2}x^3 + \frac{5}{3}x^2 - \frac{14}{9}x + \frac{4}{9}\right)$$

$$= \left[-\frac{1}{8}x^4 + \frac{5}{9}x^3 - \frac{7}{9}x^2 + \frac{4}{9}x\right]_0^2 = \frac{2}{9}.$$

Alternatif yol: $\sigma_X^2 = E\left[X^2\right] - \left\{E\left[X\right]\right\}^2 = \dfrac{2}{3} - \dfrac{4}{9} = \dfrac{2}{9}.$

(b) 2*X* + 3'ün ortalma ve varyansı nedir?

$$E[2X+3]=E[2X]+E[3]=2E[X]+3=2\times\frac{2}{3}+3=\frac{13}{3}.$$

$$E\left[(2X+3)^2\right]=E\left[4X^2+12X+9\right]=4E\left[X^2\right]+12E[X]+9$$

$$=4\times\frac{2}{3}+12\times\frac{2}{3}+9=\frac{59}{3}.$$

$$\sigma^2_{2X+3}=E\left[(2X+3)^2\right]-\left\{E[2X+3]\right\}^2=\frac{59}{3}-\left(\frac{13}{3}\right)^2=\frac{8}{9}$$

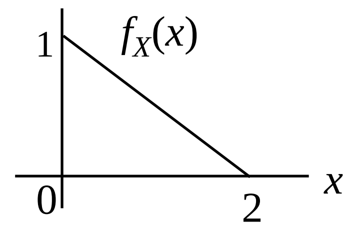
veya

$$\sigma^2_{2X+3}=\mathrm{var}[2X+3]=\mathrm{var}[2X]=4\,\mathrm{var}[X]=4\times\frac{2}{9}=\frac{8}{9}.$$

<u>Örnek</u>:

Y = 2X + 3, ifadesinde X oyf si aşağıda verilen bir r.d. ise

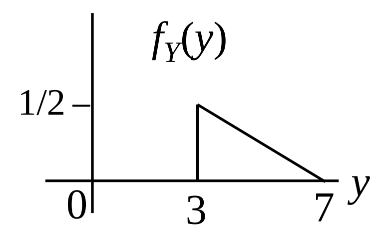
$$f_X(x) = \begin{cases} 1 - \dfrac{x}{2}, & 0 \le X \le 2 \\ 0, & \text{diğer} \end{cases}$$



(a) Y'nin oyf sini bulun.

$$f_Y(y) = \frac{1}{|dy/dx|} f_X(x)\bigg|_{x=g^{-1}(y)}$$ formülünden

ve $\left|\dfrac{dy}{dx}\right| = 2$ ile $x = \dfrac{y-3}{2}$ ifadelerinden



$$f_Y(y) = \frac{1}{2} f_X\left(\frac{y-3}{2}\right) = \frac{1}{8}(7-y), \quad 3 \le \text{y} \le 7$$

bulunur.

(b) *Y*'nin ortalamasını, ikinci momentini ve varyansını bulun.

$$m_Y = E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)dy = \frac{1}{8}\int_3^7 y\ (7-y)dy = \left[\frac{7y^2}{16} - \frac{y^3}{24}\right]_3^7 = \frac{13}{3}.$$

$$E\left[Y^2\right] = \frac{1}{8}\int_3^7 y^2\ (7-y)dy = \left[\frac{7y^3}{24} - \frac{y^4}{32}\right]_3^7 = \frac{59}{3}.$$

$$\text{var}[Y] = \sigma_Y^2 = \frac{1}{8}\int_3^7 \left(y - \frac{13}{3}\right)^2 (7-y)dy = \int_3^7 \left(-\frac{y^3}{8} + \frac{47}{24}y^2 - \frac{715}{72}y + \frac{1183}{72}\right)$$

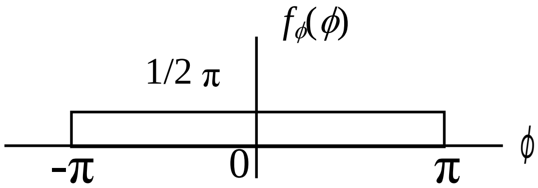
$$= \left[-\frac{1}{32}y^4 + \frac{47}{72}y^3 - \frac{715}{144}y^2 + \frac{1183}{72}y\right]_3^7 = \frac{8}{9}.$$

<u>Örnek</u>:

$X = A\cos(\omega t_0 + \phi)$ ϕ [−π, π] arasında birbiçim dağılmış

$X = g(\phi)$

$$E[X] = E_\phi[g(\phi)] = \int g(\phi) f_\phi(\phi) d\phi$$



(a) Ortalama

$$\begin{aligned}
m_X &= E[A\cos(\omega t_0 + \phi)] \\
&= A\int_{-\pi}^{\pi} \cos(\omega t_0 + \phi) \frac{1}{2\pi} d\phi \\
&= \frac{A}{2\pi} \cdot \sin(\omega t_0 + \phi) \Big|_{-\pi}^{\pi} \\
&= \frac{A}{2\pi}[\sin(\omega t_0 + \pi) - \sin(\omega t_0 - \pi)] \\
&= \frac{A}{2\pi}[\sin\omega t_0 \cos\pi + \cos\omega t_0 \underbrace{\sin\pi}_{0} - \sin\omega t_0 \cos\pi + \cos\omega t_0 \underbrace{\sin\pi}_{0}] = 0
\end{aligned}$$

(b) Varyans

$$\sigma_X^2 = E\left[A^2 \cos^2(\omega t_0 + \phi) \right]$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \cos^2(\omega t_0 + \phi) d\phi = \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \frac{1+\cos(2\omega t_0 + 2\phi)}{2} d\phi$$

$$= \frac{A^2}{4\pi}\int_{-\pi}^{\pi} d\phi + \frac{A^2}{4\pi}\int_{-\pi}^{\pi} \cos(2\omega t_0 + 2\phi) d\phi$$

$$= \frac{A^2}{4\pi} \cdot \phi \Big|_{-\pi}^{\pi} + \frac{A^2}{4\pi} \cdot \frac{\sin(2\omega t_0 + 2\phi)}{2} \Big|_{-\pi}^{\pi}$$

$$= \frac{A^2}{2} + \frac{A^2}{8\pi}\left[\sin(2\omega t_0 + 2\pi) - \sin(2\omega t_0 - 2\pi)\right]$$

$$= \frac{A^2}{2} + \frac{A^2}{8\pi}\left[\sin 2\omega t_0 - \sin 2\omega t_0\right] = \frac{A^2}{2}.$$

Örnek: Üstel r.d. $f_X(x) = \lambda e^{-\lambda x}$ $x \geq 0$

(a) Ortalama

$$m_x = \int_0^\infty x\lambda e^{-\lambda x} dx$$

$$= x\frac{\lambda e^{-\lambda x}}{-\lambda}\bigg|_0^\infty - \int_0^\infty \frac{\lambda e^{-\lambda x}}{-\lambda}$$

$$= \lim_{x\to\infty}\left[-xe^{-\lambda x}\right] + \frac{e^{-\lambda x}}{-\lambda}\bigg|_0^\infty = 0 + \frac{1}{\lambda}.$$

(b) Varyans

İlk olarak ikinci momenti bulalım

$$E\left[X^2\right]=\int_0^\infty x^2\lambda e^{-\lambda x}dx$$

$$=x^2\left.\frac{\lambda e^{-\lambda x}}{\lambda}\right|_0^\infty+\int_0^\infty\frac{2x\lambda e^{-\lambda x}}{\lambda}dx$$

$$=0+\left.\frac{2x\lambda e^{-\lambda x}}{-\lambda}\right|_0^\infty+\int_0^\infty\frac{2e^{-\lambda x}}{\lambda}dx$$

$$=0+0+2\left.\frac{e^{-\lambda x}}{\lambda^2}\right|_0^\infty=\frac{2}{\lambda^2}$$

Buradan da varyans bulunur

$$\sigma_X^2=E\left[X^2\right]-m_X^2=\frac{2}{\lambda^2}-\left[\frac{1}{\lambda}\right]^2=\frac{1}{\lambda^2}.$$

<u>Örnek</u>:

[0, ∞] aralığındaki bir I geometrik rastgele değişkeni için ortalam ve varyansı bulun:

$$f_I[i] = p(1-p)^i, \quad i = 0,1,2,\cdots,\infty$$

Ortalama

$$E \quad E[I] = \sum_{i=0}^{\infty} i\, f_I[i] = p\sum_{i=0}^{\infty} i\,(1-p)^i = p\sum_{i=0}^{\infty}(1-p)\left\{i\,(1-p)^{i-1}\right\}$$

Hatırlatma

$$i(1-p)^{i-1} = -\frac{d}{dp}\left[(1-p)^i\right]$$

buradan

$$E[I] = p(1-p)\sum_{i=0}^{\infty}\left\{-\frac{d}{dp}\left[(1-p)^i\right]\right\} = -p(1-p)\frac{d}{dp}\sum_{i=0}^{\infty}(1-p)^i$$

$$E[I] = p(1-p)\left(\frac{1}{1-(1-p)}\right) = -p(1-p)\left[-\frac{1}{p^2}\right] = \frac{1-p}{p}.$$

İkinci Moment

$$E[I^2] = \sum_{i=0}^{\infty} i^2 \; f_I[i] = p\sum_{i=0}^{\infty} i^2 (1-p)^i = p\sum_{i=0}^{\infty} i(1-p)\left\{i \; (1-p)^{i-1}\right\}$$

$$= p(1-p)\sum_{i=0}^{\infty} i\left\{-\frac{d}{dp}\left[(1-p)^i\right]\right\} = -p(1-p)\,\frac{d\sum^{\infty}\cdots}{\text{ortalama/p}}$$

$$E[I^2] = i(1-p)\frac{d}{dp}\left(\frac{1-p}{p^2}\right) = -i(1-p)\left(-\frac{2-p}{p^3}\right) = \frac{p^2-3p+2}{p^2}$$

Varyans

$$\sigma_I^2 = E[I^2] - \left\{E[I]\right\}^2 = \frac{p^2-3p+2}{p^2} - \left(\frac{1-p}{p}\right)^2 = \frac{1-p}{p^2}$$

Dönüşüm Yöntemleri

Moment-Üreten Fonksiyon (MÜF)

$$M_X(s) = E\left[e^{sX}\right] = \int_{-\infty}^{\infty} f_X(x) e^{sx} dx$$

$\Rightarrow$ oyf'nin Laplace transformu (üstelin işareti değişmiş!)

Örnek:

Üstel r.d. n,n MÜF'ünü bulun

$$M_X(s) = \int_0^{\infty} e^{-\lambda x} e^{sx} dx = e^{-(\lambda - s)x} dx$$

$$= \frac{\lambda}{-(\lambda - s)} e^{-(\lambda - s)x} \Big|_0^{\infty} = \frac{\lambda}{\lambda - s} \qquad (\mathrm{Re}[s] < \lambda)$$

n. moment (moment toremi)

$$E\left[X^n\right] = \left.\frac{d^n M_X(s)}{ds^n}\right|_{s=0}$$

<u>Örnek</u>: (üstel r.d.)

$$M_X(s) = \frac{\lambda}{\lambda - s}$$

(a) $n = 1 \Rightarrow$ $E[X] = \left.\frac{dM_X(s)}{ds}\right|_{s=0} = \left.\frac{\lambda}{(\lambda - s)^2}(-1)\right|_{s=0} = \frac{1}{\lambda}$

(b) $n = 2 \Rightarrow$ $E\left[X^2\right] = +\left.\frac{2\lambda}{(\lambda - s)^3}\right|_{s=0} = \frac{2}{\lambda^2}$

(c) Varyans

$$\sigma^2 = E\left[X^2\right] - \left(E[X]\right)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Olasılık Üreten Fonksiyon (kesikli r.d.)

$$G_I(z) = E\left[z^i\right] = \sum_{i=-\infty}^{\infty} f_I[i] z^i$$

$\Rightarrow$ OKF'nin *z*-transformu (işaret değişik!)

Örnek: Poisson r.d.

$$G_I(z) = \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} e^{-\alpha} z^i = e^{-\alpha} \sum_{i=0}^{\infty} \frac{(\alpha z)^i}{i!} = e^{\alpha(z-1)}$$

I nın negatif olmayan tam sayı değerleri için *Olasılıkları Üretir*:

$$\frac{1}{n!} \frac{d^n G_I(z)}{dz^n}\bigg|_{z=0} = \mathrm{f}_I[n] = \Pr[I = n]$$

Momentler

$$E[I] = \left.\frac{dG_I(z)}{dz}\right|_{z=1}$$

$$E[I^2] = \left.\frac{d^2G_I(z)}{dz^2}\right|_{z=1} + \left.\frac{dG_I(z)}{dz}\right|_{z=1}$$

<u>Örnek</u>: (Poisson r.d.) $G_I(z) = e^{\alpha(z-1)}$

$$E[I] = \left.\frac{dG_I(z)}{dz}\right|_{z=1} = e^{-\alpha}\alpha e^{\alpha z}\Big|_{z=1} = \alpha$$

$$\left.\frac{dG_I(z)}{dz}\right|_{z=1} = e^{-\alpha}\alpha^2 e^{\alpha z}\Big|_{z=1} = \alpha^2$$

$$\therefore \quad E\left[I^2\right] = \alpha^2 + \alpha$$

$$\sigma_I^2 = E\left[I^2\right] - \left(E[I]\right)^2 = \alpha^2 + \alpha - (\alpha)^2 = \alpha$$

<u>Örnek</u>: Geometrik r.d.

Olasılık Üreten Fonksiyon

$$G_I(z) = \sum_{i=0}^{\infty} p(1-p)^i z^i = p\sum_{i=0}^{\infty}\left((1-p)z^i\right) = p\cdot\frac{1}{p-(1-p)z}$$

Ortalama: $$E[I] = \left.\frac{dG_I(z)}{dz}\right|_{z=1} = \left.\frac{p(1-p)}{\left(1-(1-p)z\right)^2}\right|_{z=1} = \frac{1-p}{p}$$

Varyans:

$$E[I^2] = \left.\frac{d^2G_I(z)}{dz^2}\right|_{z=1} + \left.\frac{dG_I(z)}{dz}\right|_{z=1} = \left.\frac{d}{dz}[\quad]\right|_{z=1} + \frac{1-p}{p}$$

$$= \left.\frac{2p(1-p)^2}{\left(1-(1-p)z\right)^3}\right|_{z=1} + \frac{1-p}{p} = 2\left(\frac{1-p}{p}\right)^2 + \frac{1-p}{p}$$

$$\sigma_I^2 = E[I^2] - \left(E[I]\right)^2 = 2\left(\frac{1-p}{p}\right)^2 + \frac{1-p}{p} - \left(\frac{1-p}{p}\right)^2 = \left(\frac{1-p}{p}\right)^2 + \frac{1-p}{p} = \frac{1-p}{p^2}$$

Özet ve Bağıntılar

MÜF

$M(s)$

$M(jw) \rightarrow$ Karakteristik Fonksiyon

OÜF

$G(z)$

Kesikli rastgele değişken için: $M(s) = G(z)\big|_{z=e^s}$