MARKOV ANALYSIS

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CONTENT

• Definition
• Assumption
• Algebraic determination
• Literature on Markov analysis
What is Markov analysis?

- Markov analysis is a descriptive technique that results in probabilistic information.
- Markov analysis is a system exhibiting probabilistic movement from one state to another, over time.
- Analyze the future probabilities based on current and known probabilities.
- Markov analysis is a process based on conditional probability function.
• Frequently used in law, marketing, education, medical services, income distribution, migration and meteorology

• Beneficial when analyzing market share, debt estimation, planning machine use time
For example, market share of two rival producers are 40% and 60%, respectively in the starting situation.

After 2 months, they become 45% and 55%.

For estimating the final situation, transition probability should be known.
The transition matrix includes the transition probabilities for each state of nature.
Assumptions

1. The probabilities of moving from a state to all others sum to one,
2. The probabilities apply to all system participants,
3. The probabilities are constant over time
4. The states are independent over time
Mathematical presentation

\[ \pi (i) = \text{vector of state probabilities for } i \text{th period} \]
\[ = (\pi_1, \pi_2, \ldots, \pi_n) \]

\( n = \text{the number of state} \)

\( \pi_1, \pi_2, \ldots, \pi_n = \text{probabilities of state} \)
For example;

• Two state for machine:

State 1: work

State 2: out of work

\[ \pi(1) = (\pi_1, \pi_2) = (1,0) \]

\[ \pi(1) = Initial \ starting \ state \]

\[ \pi_1 = state \ 1 \ (work) \]

\[ \pi_2 = state \ 2 \ (out \ of \ work) \]
For example;

100 thousand people live in a county
40 thousand = American Food Store,
30 thousand = Food Mart,
30 thousand = Atlas Foods müşteri

**Shopping probabilities;**
State 1= American Food Store = 0,40
State 2= Food Mart = 0,30
State 3= Atlas Foods = 0,30
Vector of state probabilities
\[ \pi (1) = (0.4, 0.3, 0.3) \]

\[ \pi (1) = \text{vector of state of 3 markets in first period} \]

\[ \pi_1 = 0.40 = \text{state 1 (the probability of shopping from American Food Store)} \]

\[ \pi_2 = 0.30 = \text{state 2 (the probability of shopping from Food Mart)} \]

\[ \pi_3 = 0.30 = \text{state 3 (the probability of shopping from Atlas Food)} \]
Transitivity matrix \((P)\)
Transition probability matrix for 3 markets

\[ P = \begin{bmatrix}
0.8 & 0.1 & 0.1 \\
0.1 & 0.7 & 0.2 \\
0.2 & 0.2 & 0.6
\end{bmatrix} \]
Örneğin:

- $P_{11} = 0.8$ = ilk dönem 1. marketten alışveriş yapanların 2. dönemde de buradan alışveriş yapma olasılıkları

- $P_{21} = 0.1$ = ilk dönem 2. marketten alışveriş yapanların 2. dönemde 1. marketten alışveriş yapma olasılıkları

- $P_{32} = 0.2$ = ilk dönem 3. marketten alışveriş yapanların 2. dönemde 2. marketten alışveriş yapma olasılıkları
ATTENTION:

The sum of the probabilities of each row must be 1.
Forecast for the future,

\[ \pi(1) = \pi(0)P \]

Where 0 is initial starting state and 1 is future situation.

Forecasting n and n+1;

\[ \pi(n + 1) = \pi(n)P \]
Market share for 3 markets in second period;

\[
\pi(1) = \pi(0)P
\]

\[
= (0.4, 0.3, 0.3) \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}
\]

\[
= [(0.4)(0.8) + (0.3)(0.1) + (0.3)(0.2), (0.4)(0.1) + (0.3)(0.7) + (0.3)(0.2), (0.4)(0.1) + (0.3)(0.2) + (0.3)(0.6)]
\]

\[
= (0.41, 0.31, 0.28)
\]
Steady-state probabilities

• If machine works good in first and second month, it is good by 80% (bad by 20%) in a following month.

• If machine works not good in first and second month, the probability of working bad is 90% (10% for being good in a following month.

• State 1: machine in good condition and work
• State 2: machine in good condition and out of work
Transition matrix;

\[ P = \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix} \]

Probability of working of machine next month;

\[
\pi(1) = \pi(0)P
\]

\[
= (1, 0) \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}
\]

\[
= [(1)(0.8) + (0)(0.1), (1)(0.2) + (0)(0.9)]
\]

\[
= (0.8, 0.2)
\]
Following month;

\[
\pi(2) = \pi(1)P \\
= (0.8, 0.2) \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix} \\
= [(0.8)(0.8) + (0.2)(0.1), (0.8)(0.2) + (0.2)(0.9)] \\
= (0.66, 0.34)
\]

The probability of working in third month is 66%.
Results for 15 months

<table>
<thead>
<tr>
<th></th>
<th>0.0000000</th>
<th>0.0000000</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>2</td>
<td>0.800000</td>
<td>0.200000</td>
</tr>
<tr>
<td>3</td>
<td>0.660000</td>
<td>0.340000</td>
</tr>
<tr>
<td>4</td>
<td>0.562000</td>
<td>0.438000</td>
</tr>
<tr>
<td>5</td>
<td>0.493400</td>
<td>0.506600</td>
</tr>
<tr>
<td>6</td>
<td>0.445380</td>
<td>0.554620</td>
</tr>
<tr>
<td>7</td>
<td>0.411766</td>
<td>0.588234</td>
</tr>
<tr>
<td>8</td>
<td>0.388236</td>
<td>0.611763</td>
</tr>
<tr>
<td>9</td>
<td>0.371765</td>
<td>0.628234</td>
</tr>
<tr>
<td>10</td>
<td>0.360235</td>
<td>0.639754</td>
</tr>
<tr>
<td>11</td>
<td>0.352165</td>
<td>0.647834</td>
</tr>
<tr>
<td>12</td>
<td>0.346515</td>
<td>0.653484</td>
</tr>
<tr>
<td>13</td>
<td>0.342560</td>
<td>0.657439</td>
</tr>
<tr>
<td>14</td>
<td>0.339792</td>
<td>0.660207</td>
</tr>
<tr>
<td>15</td>
<td>0.337854</td>
<td>0.662145</td>
</tr>
</tbody>
</table>
Where;

\[ \pi = \pi P \]

\[
\begin{pmatrix}
\pi_1 \\
\pi_2
\end{pmatrix}
= 
\begin{pmatrix}
\pi_1 \\
\pi_2
\end{pmatrix}
\begin{bmatrix}
0.8 & 0.2 \\
0.1 & 0.9
\end{bmatrix}
\]

\[
\begin{pmatrix}
\pi_1 \\
\pi_2
\end{pmatrix}
= 
\begin{pmatrix}
\pi_1(0.8) + \pi_2(0.1) \\
\pi_1(0.2) + \pi_2(0.9)
\end{pmatrix}
\]
Values in equation;

\[ \pi_2 = 2\pi_1 \]

\[ \pi_1 + \pi_2 = 1 \]

\[ \pi_1 + 2\pi_1 = 1 \]

\[ 3\pi_1 = 1 \]

\[ \pi_1 = \frac{1}{3} = 0.333333333 \]

\[ \pi_2 = \frac{2}{3} = 0.666666667 \]
In long term;

- Working in good condition (state 1)
  \[= \frac{1}{3} (0,33)\]

- Working in bad condition (state 2)
  \[= \frac{2}{3} (0,67)\]

- **TRANSITION MATRIX MEASURES UP FOR DETERMINING STEADY-STATE PROBABILITIES.**
Sometimes we have no chance moving from one state to another.
Example

System according to costumers and debt and 4 different state

State 1: Pull in a cash all debt
State 2: bad debt, delayed payment (more than 3 months)
State 3: delayed payment, less than 1 months
State 4: delayed payment, between 1 months and 3 months
In Markov process;

• The probability of pulling in cash all debt in state 1 is 100%.

• The probability of staying state 4 is 1.
• There is no chance to change situation in state 2 and 3.

• If there is no chance to change situation, the probability of being same situation is accepted by %100.

• Table presents other probabilities:
<table>
<thead>
<tr>
<th>Current months</th>
<th>Paid</th>
<th>Bad debt</th>
<th>&lt;1 AY</th>
<th>1-3 AY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paid</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Bad debt</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>&lt;1 AY</td>
<td>0,6</td>
<td>0</td>
<td>0,2</td>
<td>0,2</td>
</tr>
<tr>
<td>1-3 AY</td>
<td>0,4</td>
<td>0,1</td>
<td>0,3</td>
<td>0,2</td>
</tr>
</tbody>
</table>
• Transition matrix;

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0.6 & 0 & 0.2 & 0.2 \\
0.4 & 0.1 & 0.3 & 0.2
\end{bmatrix}
\]

• In long term, each customer absolutely will take place one of the state.
• How many people will group in these state?

• We need the amount of total debt

• Constructing fundamental matrix.

• Transition matrix divided into parts.
Thus;

\[
P = \begin{bmatrix}
I & 0 \\
\downarrow & \downarrow \\
1 & 0 \\
0 & 1 \\
0.6 & 0 \\
0.4 & 0.1
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
0.2 \\
0.3 \\
0.2
\end{array}
\end{bmatrix}
\]

\[
I = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
0.6 & 0 \\
0.4 & 0.1
\end{bmatrix}
\]

\[
0 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0.2 & 0.2 \\
0.3 & 0.2
\end{bmatrix}
\]
Obtaining fundamental matrix;

\[ F = (I - B)^{-1} \]

\[ F = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} \right)^{-1} \]

\[ F = \begin{bmatrix} 0.8 & -0.2 \\ -0.3 & 0.8 \end{bmatrix}^{-1} \]

\[ F = \begin{bmatrix} 0.8 & -0.2 \\ -0.3 & 0.8 \end{bmatrix} \begin{bmatrix} \frac{0.8}{0.58} & \frac{-(-0.2)}{0.58} \\ \frac{-(-0.3)}{0.58} & \frac{0.8}{0.58} \end{bmatrix} = \begin{bmatrix} 1.38 & 0.34 \\ 0.52 & 1.38 \end{bmatrix} \]
• Calculation of bad debt using fundamental matrix

\[ FA = \begin{bmatrix} 1.38 & 0.34 \\ 0.52 & 1.38 \end{bmatrix} \times \begin{bmatrix} 0.6 & 0 \\ 0.4 & 0.1 \end{bmatrix} \]

\[ FA = \begin{bmatrix} 0.97 & 0.03 \\ 0.86 & 0.14 \end{bmatrix} \]

• Fundamental matrix represents the probability of no chancing.
The probability of delayed payment, less than 1 months (0.97 paid, 0.03 bad debt)

\[ FA = \begin{bmatrix} 0.97 & 0.03 \\ 0.86 & 0.14 \end{bmatrix} \]

The probability of delayed payment, between 1 and 3 months (0.86 paid, 0.14 bad debt)
\[ M = (M_1, M_2, M_3, \ldots, M_n) \]

n = number of state
M_1 = quantity in state 1
M_2 = quantity in state 2
M_n = quantity in state n
Less than 1 months = 2000$, 1-3 months = 5000$;

M matrix \[ M = (2000, 5000) \]

paid/bad debt quantity = M*FA

\[
\begin{bmatrix}
0.97 & 0.03 \\
0.86 & 0.14 \\
\end{bmatrix}
\]

\[
= (6240, 760)
\]

7000$ (2000+5000) paid = 6240$, bad debt=760$
ÖRNEK

Bursa’da üretilen Sütaş ve Eker sütleri aynı pazarda satılmaktadır. Son Sütaş ürünü alan bir kişinin gelecekte aynı ürünü alma olasılığı %90 ve son Eker ürünü alan bir kişinin aynı ürünün alınma olasılığı %80’dir. Süt tüketen haftalık müşteri sayısı 15.000 kişidir. Her müşterihaftada bir kez 2 şişe süt almaktadır. Bir şişe sütün Şütaş’a maliyeti 0,6$, satış fiyatı 0,8$’dir. Bir reklam şirketi Şütaş’a şöyle bir teklifte bulunur. Yılda 20.000$ ödendiğinde Sütaş müşterisinin bir sonraki hafta Eker sütü alma olasılığının %10’dan %5’e düşüreleeceği garanti edilir.
Buna göre;

1. Şimdi Eker süt alan bir kişinin, iki alım sonra Sütaş alma olasılığı nedir?

2. Şimdi Sütaş alan bir kişinin üç alım sonra Sütaş alma olasılığı nedir?

3. Sütaş reklam şirketinin teklifini kabul etmeli midir?
1. Geçiş olasılığı matrisi

\[ P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \]

Gelecek 2 alım için;

\[ P^2 = P \times P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \]

\[ = \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.06 \end{bmatrix} \]

Yani, Eker sütü alan bir kişinin iki alım sonra Sütaş alma alma alma olasılığı 0,34’dür.
2. Geçiş olasılığı matrisi

\[
P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}
\]

Gelecek 3 alım için;

\[
P^3 = P \times P \times P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.06 \end{bmatrix} = \begin{bmatrix} 0.781 & 0.219 \\ 0.438 & 0.562 \end{bmatrix}
\]

Yani, Sütaş sütü alan bir kişinin üç alım sonra Sütaş alma alma alma olasılığı 0,781’dir.
3. Denge pazar payı;

\[ P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \]

\[ [q_1 q_2] = [q_1 q_2] \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \]

Denklemde değerler ile;

\[ q_1 = \frac{2}{3} \]
\[ q_2 = \frac{1}{3} \] elde edilir.

Yani Sütaş’ın pazar payı 2/3 olarak bulunur.
Her müşteri haftada 2 şişe süt alırsa, bir yıldaki süt satışı;
52* (15.000*2)=1.560.000 şişe

Sütaş'ın payı;
2/3*(1.560.000)=1.040.000 şişedir.

Şirketin karı;
(0,8-0,6)*1.040.000=208.000$ olarak bulunur.
Reklem şirketi P matrisini aşağıdaki gibi değiştirmeyi teklif etmektedir.

\[ P = \begin{bmatrix} 0.95 & 0.05 \\ 0.20 & 0.80 \end{bmatrix} \]

Denklemde değerler ile;
\( q_1 = 0.8 \)
\( q_2 = 0.2 \) elde edilir.

Bu durumda Sütaş’ın karı;

\[
1.560.000 \times 0.8 \times 0.2 = 249.600 \]

\( 249.600 - 20.000 = 229.600 \) $ kar elde edilir.

Bu durumda Sütaş, karını **21.600$** (229.600-208.000) arttıracağından bu teklifi kabul etmelidir.