NON-LINEAR PROGRAMMING (NLP)

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Similarities & differences

Characteristics of NLP

NLP models

- One Variable NLP
- Multi Variable NLP
- Unlimited NLP
- Limited NLP

Basic concept

Case studies
Similarities & differences

• Linear programming (LP)
  – Linear target function + linear constraints
  – Continuous variable

• Integer programming (IP)
  – Linear target function + linear constraints
  – Discrete variable

• Non-linear programming (NLP)
  – The objective is linear but constraints are non-linear
  – Non-linear objectives and linear constraints
  – Non-linear objectives and constraints
Characteristics of NLP

- Solution is difficult
- Solution may tie initial point.
- Initial point is subjective.
• NLP forms:
  – Non-linear objectives
    \[ \min (x - 3)^2 \]
    \[ s.t. \; x < 3 \]
  – Non-linear constraints
    \[ \min \; x_1 + x_2 \]
    \[ s.t. \; x_1^2 + x_2^2 \leq 4 \]
  – Non-linear objectives and constraints
    \[ \min \; (x_1 - 2)^2 + x_2 \]
    \[ s.t. \; x_1^2 + x_2^2 \leq 4 \]

➤ The main problem is algorithm selection
• Constructing model is required for solving optimization problem. Mathematical model is based on determination of variables and defining their functional relationship.

• 3 components of mathematical model

  – Objective function,
  – Constraints
  – Non-negativity restriction
• Linear programming, integer programming and goal programming assume that objective function and constraints is linear. Not include non-linear expressions such as $1/X_2$, $\log X_3$

• NLP procedures does not produce optimum solution every time in contrary with linear programming (Render ve ark, 2012).
Objective function in NLP

- \( g_i(x) \leq b_i \) constraints

- \( z = f(x) \) objective function

We find the vector of \( x = (x_1, x_2, \ldots, x_n) \), which produce optimum solution for objective
NLP models

• Associated with the number of variable
  – One variable or multivariate,

  ▪ Associated with the presence of restrictions
  – restricted or unrestricted
NLP models

One variable models

- Limited Model: Constraints: equality
- Unrestricted Model: Constraints: inequality

Multivariate models

- Limited Model: Constraints: equality
- Unrestricted Model: Constraints: inequality
Basic concepts

• Increasing and decreasing function

\[ y = f(x) \]

\( x_1 \) and \( x_2 \) is a random figure

If the function is \( f(x_1) < f(x_2) \) when inequality is \( x_1 < x_2 \), then function is called increasing.

If the function is \( f(x_1) > f(x_2) \) when inequality is \( x_1 < x_2 \), then function is called decreasing.
Increasing and decreasing function

Increasing function

Decreasing function
– Local Maximum and Minimum

\[ f(x)'' > f(x)' \]
\[ f(x)'' < f(x)' \quad \text{local min. or max} \]

– Gradient of function

Gradient function of \( Z=f(x_1, x_2, ..., x_n) \) is vector of first derivatives.
Hessian Matrix

• Hessian matrix is a $n \times n$ matrix of second partial derivative of $f(x_1,x_2,\ldots,x_n)$ function

$$H_f = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{n \times n}$$
✓ In LP, solution region is convex set and optimum solution is one of the corner point.

✓ In NLP, it is not necessary that optimum solution is one of the corner point.
Concave function

Convex function
Example:

1) Is \( f(x) = \sqrt{x} \) function whether convex or concave at \( [S = 0, \infty) \)?

✓ Since line between two points take place under the curve, the function is concave.
2) Is $f(x)=x^3$ function whether convex or concave at $S = \mathbb{R}^1= (-\infty, \infty)$?

Neither convex nor concave function
3) Is $f(x) = 3x - 3$ function whether convex or concave at $S = \mathbb{R}^1 = (-\infty, \infty)$?

Both convex and concave function
One variable unrestricted NLP

• Deal with finding maximum or minimum point of one variable function with no restriction
Multivariate unrestricted NLP

• Deal with finding optimum point \((x_1^*, x_2^*, \ldots, x_n^*)\) that is maximum or minimum point of multivariate function of \(f(x_1, x_2, \ldots, x_n)\) with no restriction

• Demand function explained by price, preference, income, etc. Is an example for multivariate unrestricted NLP.
Restricted NLP

- Examining the colinearity among \( x_1, x_2, \ldots, x_n \) that is variables of multivariate function of \( f(x_1, x_2, \ldots, x_n) \)
- \( f(x_1, x_2, \ldots, x_n) \) function,
- \( g_1(x_1, x_2, \ldots, x_n) = b_1 \)
- \( g_2(x_1, x_2, \ldots, x_n) = b_2 \) restrictions
- \( g_m(x_1, x_2, \ldots, x_n) = b_m \)

We are looking for point that is maximum or minimum under upper restrictions.
Solution

• Lagrange function

\[ L(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m) = f(x_1, x_2, \ldots, x_n) + \sum (b_i - g_i(x_1, x_2, \ldots, x_n)) \]
Example

Minimize \[ C = (x_1 - 4)^2 + (x_2 - 4)^2 \]
subject to \[ 2x_1 + 3x_2 \geq 6 \]
\[ -3x_1 - 2x_2 \geq -12 \]
and \[ x_1, x_2 \geq 0 \]

The Lagrangian is:
\[ Z = (x_1 - 4)^2 + (x_2 - 4)^2 + \lambda_1 (6 - 2x_1 - 3x_2) + \lambda_2 (-12 + 3x_1 + 2x_2) \]

Kuhn Tucker Conditions:
\[ Z_{x_1} = 2(x_1 - 4) - 2\lambda_1 + 3\lambda_2 \geq 0 \quad x \geq 0 \quad \text{and} \quad x_1Z_{x_1} = 0 \]
\[ Z_{x_2} = 2(x_2 - 4) - 3\lambda_1 + 2\lambda_2 \geq 0 \quad y \geq 0 \quad \text{and} \quad x_2Z_{x_2} = 0 \]
\[ Z_{\lambda_1} = 6 - 2x_1 - 3x_2 \leq 0 \quad \lambda_1 \geq 0 \quad \text{and} \quad \lambda_1Z_{\lambda_1} = 0 \]
\[ Z_{\lambda_2} = -12 + 3x_1 + 2x_2 \leq 0 \quad \lambda_2 \geq 0 \quad \text{and} \quad \lambda_2Z_{\lambda_2} = 0 \]

Solution: \[ x_1 = \frac{28}{13}, \quad x_2 = \frac{36}{13}, \quad \lambda_1 = 0, \quad \lambda_2 = \frac{16}{13} \]
Non-linear objective function and linear constraints
Case of non-linear objective function and linear constraints

Great Western Appliance firm sell toast machine of Mikrotoaster ($X_1$) and Self-Clean Toaster Oven ($X_2$). Firm gain net profit by $28 per toaster. Profit function is $21X_2 + 0.25X_2^2$

Objective function is non-linear:

Maximum profit = $28X_1 + 21X_2 + 0.25X_2^2$

There are two linear constraints

$X_1 + X_2 \leq 1,000$ (production capacity)

$0.5X_1 + 0.4X_2 \leq 500$ (term of sale)

$X_1, X_2 \geq 0$

(Source: Render ve ark., 2012)
2. Quadratic programming

- Objective function includes terms such as $0.25x^2_2$ and when the restrictions is linear

- Quadratic programming problems can be solved by using adjusted simplex algorithms.

(Source: Render and ark., 2012)
EXAMPLES
WIN QSB for NLP
WIN QSB EKRANINI TANIYALIM

- Defining new problem
- Open existing file
- Closing Win QSB
- Calculator
- Clock
- Help
Unrestricted NLP example

• minimum $x(\sin(3.14159x))$
• $0 \leq x \leq 6$

• We have one non-linear expression such as $\sin(3.14159x)$ for minimizing objective function
For unrestricted case, enter “0”
Unrestricted NLP example
Attention!

Interval of $X_1$

Graphical solution
x=3.5287
Objective function= -3.5144.
Example 2:

**Objective function:**

maximum \(2x_1 + x_2 - 5\log_e(x_1)\sin(x_2)\)

**Constraints**

\[x_1x_2 \leq 10\]
\[|x_1 - x_2| \leq 2\]
\[0.1 \leq x_1 \leq 5\]
\[0.1 \leq x_2 \leq 3\]
### Maximize

\[ 2x_1 + x_2 - 5 \log(x_1) \sin(x_2) \]

### Constraints

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>[ x_1 \cdot x_2 \leq 10 ]</td>
</tr>
<tr>
<td>C2</td>
<td>[ \text{abs}(x_1 - x_2) \leq 2 ]</td>
</tr>
</tbody>
</table>

### Variable Bounds

- \( x_1 \geq 0.1, \leq 5 \)
- \( x_2 \geq 0.1, \leq 3 \)

**NLP**
To solve a nonlinear problem, the search methods are employed. Refer to the help file for the specific method. The initial solution, i.e., starting point, and some parameters are required for the search methods and usually affect the solution efficiency. In general, the default setup works effectively. Enter the parameters on the left and the initial solution and bounds on the right.

Maximum runtime in seconds: 3600

Variable | Lower Bound | Upper Bound | Initial Solution
---|---|---|---
X1 | 0.1 | 5 | 0
X2 | 0.1 | 3 | 0

Stopping tolerance (delta): 1.0E-4

Starting penalty parameter (u): 100

Penalty multiplier (beta): 10

Penalty power (p > 1): 2

OK Cancel Default Print Help
\[ x_1 = 3.3340 \text{ ve } x_2 = 2.9997 \]

Objective function = 8.8166
Non-linear function and constraints
• In medium size hospital having 200-400 patient bed, Hospicare Corporation, annual profit depend on number of patient $(X_1)$ ve number of patient surgeon $(X_2)$ bağlıdır.

• Non-linear objective function for Hospicare:
  \[ 13X_1 + 6X_1 X_2 + 5X_2 + 1/X_2 \]

**Constraints:**

• \[ 2X_1^2 + 4X_2 \leq 90 \] (nursery capacity)
• \[ X_1 + X_2^3 \leq 75 \] (X-ray capacity)
• \[ 8X_1 - 2X_2 \leq 61 \] (marketing budget)
<table>
<thead>
<tr>
<th>Variable</th>
<th>Constraint</th>
<th>Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>X1</td>
<td>=0, ≤6</td>
<td></td>
</tr>
<tr>
<td>X2</td>
<td>=0, ≤4</td>
<td></td>
</tr>
<tr>
<td>OBJ</td>
<td>Maximize</td>
<td></td>
</tr>
<tr>
<td>C1</td>
<td>2^2 + 4^2</td>
<td>≤90</td>
</tr>
<tr>
<td>C2</td>
<td>X1 + X2^3</td>
<td>≤75</td>
</tr>
<tr>
<td>C3</td>
<td>8^2 + 2^2</td>
<td>≤61</td>
</tr>
<tr>
<td>C4</td>
<td>X1 + X2</td>
<td>≤400</td>
</tr>
</tbody>
</table>

The objective function is to maximize:

\[ 13X_1 + 6X_1X_2 + 5X_2 + 1X_2^2 \]
To solve a nonlinear problem, the search methods are employed. Refer to the help file for the specific method. The initial solution, i.e., starting point, and some parameters are required for the search methods and usually affect the solution efficiency. In general, the default setup works effectively. Enter the parameters on the left and the initial solution and bounds on the right.

Maximum run time in seconds
3600

Stopping tolerance (delta)
1.00E-4

Starting penalty parameter (u)
100

Penalty multiplier (beta)
10

Penalty power (p, >1)
2

<table>
<thead>
<tr>
<th>Variable</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>Initial Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>X1</td>
<td>0</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>X2</td>
<td>0</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>
### Solution Summary for hospicare

<table>
<thead>
<tr>
<th>Date</th>
<th>Decision Variable</th>
<th>Solution Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>04-24-2012</td>
<td>X1</td>
<td>5.9995</td>
</tr>
<tr>
<td></td>
<td>X2</td>
<td>3.9993</td>
</tr>
<tr>
<td>Maximized</td>
<td>Objective Function</td>
<td>242.2024</td>
</tr>
</tbody>
</table>
Example:
Pickens Memorial Hospital

✓ Patient demand exceeds capacity of hospital

Objective: Maximum profit
Decision variables

\[ M = \text{number of served patient} \]
\[ S = \text{number of served patient for surgery} \]
\[ P = \text{number of served child patient} \]

Profit function

When increasing patient, profit increasing non-linearly.
Constraints

- Hospital capacity: Total 200 patient
- X-ray capacity: 560 x-rays per week
- Marketing budget: $1000 per week
- Laboratory capacity: 140 hours per week
Objective function
Max $45M + 2M^2 + 70S + 3S^2 + 2MS + 60P + 3P^2$

Constraints:

$M + S + P \leq 200$ (patient capacity)
$M + 3S + P \leq 560$ (x-ray capacity)
$3M + 5S + 3.5P \leq 1000$ (budget $\$$)
$(0.2+0.001M)(3M+3S+3P) \leq 140$ (lab hour)

$M, S, P \geq 0$
<table>
<thead>
<tr>
<th>Constraints</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>&gt;=0, &lt;=M</td>
</tr>
<tr>
<td>C2</td>
<td>&gt;=0, &lt;=M</td>
</tr>
<tr>
<td>C3</td>
<td>&gt;=0, &lt;=M</td>
</tr>
<tr>
<td>C4</td>
<td>&gt;=0, &lt;=M</td>
</tr>
<tr>
<td>X1</td>
<td>&gt;=0, &lt;=M</td>
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<tr>
<td>X2</td>
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<tr>
<td>X3</td>
<td>&gt;=0, &lt;=M</td>
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<tr>
<td>Constraint</td>
<td>Equation</td>
</tr>
<tr>
<td>-----------</td>
<td>----------</td>
</tr>
<tr>
<td>C1</td>
<td>$x_1 + x_2 + x_3 \leq 200$</td>
</tr>
<tr>
<td>C2</td>
<td>$x_1 + 3x_2 + x_3 \leq 560$</td>
</tr>
<tr>
<td>C3</td>
<td>$3x_1 + 5x_2 + 3.5x_3 \leq 1000$</td>
</tr>
<tr>
<td>C4</td>
<td>$(0.2 + 0.001x_1)(3x_1 + 3x_2 + 3x_3) \leq 140$</td>
</tr>
</tbody>
</table>

Variables:
- $x_1 \geq 0, \leq 200$ (Constraint C1)
- $x_2 \geq 0, \leq 200$ (Constraint C2)
- $x_3 \geq 0, \leq 200$ (Constraint C3)
To solve a nonlinear problem, the search methods are employed. Refer to the help file for the specific method. The initial solution, i.e., starting point, and some parameters are required for the search methods and usually affect the solution efficiency. In general, the default setup works effectively. Enter the parameters on the left and the initial solution and bounds on the right.

Maximum run time in seconds 3600

Stopping tolerance (delta)
1.00E-4

Starting penalty parameter (u) 100

Penalty multiplier (beta) 10

Penalty power [p, >1] 2
Objective function
Max 45M + 2M^2 + 70S + 3S^2 + 2MS + 60P + 3P^2

Constraints:
M + S + P ≤ 200 (patient capacity)
M + 3S + P ≤ 560 (x-ray capacity)
3M + 5S + 3.5P ≤ 1000 (budget $)
(0.2+0.001M)(3M+3S+3P) ≤ 140 (lab hour)
M, S, P ≥ 0