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14. Hafta

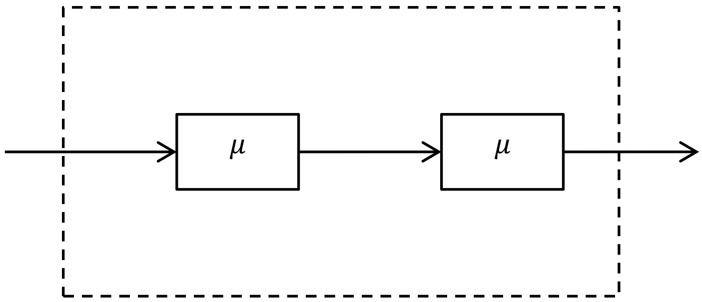
Faz dağılımları ve Faz tipi kuyruk modelleri

Üstel dağılım kuyruk modellerinde çok yaygın olarak kullanılmaktadır. Bunun nedeni üstel dağılımın belleksizlik özelliğinden kaynaklı matematiksel çözülebilirliğidir. Ancak üstel dağılımın tek başına yetersiz kaldığı süreçleri modellemek için daha genel dağılımlara ihtiyaç duyulmaktadır. Bu bağlamda faz tipi (ya da aşamalı) dağılımlar karşımıza çıkar. Faz tipi dağılımlar, bir dizi ardışık dağılımdan oluşurlar.

Erlang−2 Dağılımı

Alınan hizmet üstel dağılımlı ardışık iki faz ile veriliyor. Erlang−2 dağılımını şu örnek ile açıklayalım: iki fazdan (aşamadan) oluşan bir sözlü sınavda aday önce 1.fazda sınava alınır ve daha sonra 2.faza geçer ve sınav tamamlanmış olur. Ancak burada bir adayın sınavı tamamlanmadan bir başka aday sınava alınamaz, yani aynı anda iki aday sınavda bulunamaz.

Her bir fazdaki hizmet süresi $Y \sim \exp(\mu)$ olmak üzere, hizmet tesisi aşağıdaki gibidir.



Y'nin olasılık yoğunluk fonksiyonu,

$$f_Y(y) = \mu e^{-\mu y}\ , \ \ y \geq 0$$

ve toplam hizmet süresi $X = Y + Y$ olsun. Bu durumda iki bağımsız tesadüfi değişkenin konvülüsyon formülünden,

$$f_X(x) = \int_{-\infty}^{\infty} f_Y(y)\, f_X(x-y) dy$$

$$= \int_0^x \mu e^{-\mu y}\, \mu e^{-\mu(x-y)} dy$$

$$= \mu^2 e^{-\mu x} \int_0^x dy$$

$$= f_X(x) = \begin{cases} \mu^2 x e^{-\mu x}, & x > 0 \\ 0, & x < 0 \end{cases}$$

Burada $X \sim \text{Erlang}(2, \mu)$ veya $X \sim E_2$ ile gösterilir. X'in dağılım fonksiyonu da,

$$F_X(x) = \begin{cases} 1 & x \to \infty \\ 1 - e^{-\mu x}(1 + \mu x), & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Laplace Dönüşümü

$$L_X(s) = \int_0^{\infty} e^{-sx} f_X(x) dx$$

$$= \int_0^{\infty} e^{-sx} \mu^2 x e^{-\mu x} dx$$

$$= \mu^2 \int_0^{\infty} x e^{-(s+\mu)x} dx$$

$$= -\frac{\mu^2}{(s+\mu)} x e^{-(s+\mu)x} \Big|_0^{\infty} + \frac{\mu^2}{(s+\mu)} \int_0^{\infty} e^{-(s+\mu)x} dx$$

$$= \left(\frac{\mu}{s+\mu}\right)^2$$

Beklenen Değer ve Varyans

$$L_X(s) = \left(\frac{\mu}{s+\mu}\right)^2$$

Beklenen değer ve k −ıncı mertebeden yüksek momentler Laplace dönüşümü yardımıyla,

$$E(X) = -\frac{d}{ds} L_X(s)\bigg|_{s=0} = \frac{2}{\mu}$$

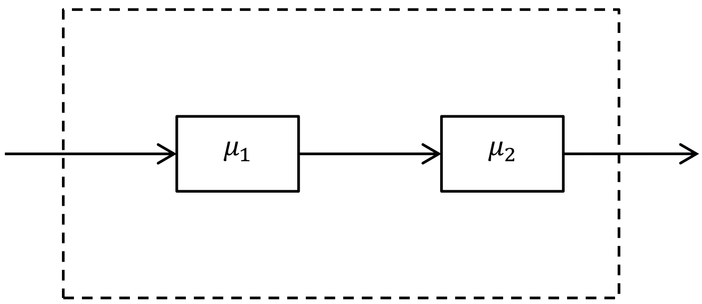
$$E(X^2) = -\frac{d^2}{ds^2} L_X(s)\bigg|_{s=0} = \frac{\mu + 2}{\mu^2}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{3}{\mu^2} - \left(\frac{1}{\mu}\right)^2 = \frac{2}{\mu^2}$$

Hipo-üstel Dağılım

Erlang−2 dağılımında her fazın aynı olan ortalama μ parametresini birbirlerinden farklı olacak şekilde μ_1 ve μ_2 olarak değiştirirsek Hipo-üstel dağılım elde edilmiş olur. Yani i −inci fazın üstel ortalama hizmet süresi μ_i olacaktır. Şekil 2 ile gösterilen bu yeni dağılıma hipo-üstel dağılım denir.



İlk önce üstel dağılımlı iki fazın olduğu durumu göz önüne alalım. Y_1 ve Y_2, parametreleri sırası ile μ_1 ve μ_2 olan üstel dağılımlı iki tesadüfi değişken ve bunların olasılık yoğunluk fonksiyonları da sırasıyla,

$$F_{Y_1}(y) = \mu_1 e^{-\mu_1 y}\,, \quad y \geq 0,$$

$$F_{Y_2}(y) = \mu_2 e^{-\mu_2 y}\,, \quad y \geq 0$$

olsunlar. $X = Y_1 + Y_2$ olsun. Burada $X \sim Hyp_2$ ile gösterilir.

Bu taktirde iki üstel dağılımın konvülüsyon formülünden X tesadüfi değişkenin olasılık yoğunluk fonksiyonu,

$$f_X(x) = \int_{-\infty}^{\infty} f_{Y_1}(y) f_{Y_2}(x-y)dy$$

$$= \int_0^x \mu_1 e^{-\mu_1 y}\, \mu_2 e^{-\mu_2 (x-y)} dy$$

$$= \mu_1 \mu_2 e^{-\mu_2 x} \int_0^x e^{-(\mu_1 - \mu_2) y} dy$$

$$= \frac{\mu_1 \mu_2}{\mu_1 - \mu_2} (e^{-\mu_2 x} - e^{-\mu_1 x}), \quad x \geq 0.$$

olarak bulunur. Dağılım fonksiyonu da,

$$F_X(x) = \begin{cases} 1, & x \to \infty \\ 1 - \dfrac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1 x} + \dfrac{\mu_1}{\mu_2 - \mu_1} e^{-\mu_2 x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Laplace dönüşümü ise,

$$L_X(s) = E(e^{-sx}) = E(e^{-Y_1}) E(e^{-Y_2})$$

olup, beklenen değer ve de varyans

$$E(X) = \frac{1}{\mu_1} + \frac{1}{\mu_2}, \qquad Var(X) = \frac{1}{{\mu_1}^2} + \frac{1}{{\mu_2}^2}$$

bulunur.

Örnek. Bir X tesadüfi değişkenin, parametreleri sırasıyla $\mu_1 = 1$ ve $\mu_2 = 2$ olan ardışık iki tane üstel fazdan oluştuğunu kabul edelim. Bu taktirde X'in olasılık fonksiyonunu, beklenen değerini ve varyansını bulalım.

Çözüm. X'in beklenen değeri, her fazın beklenen değerlerinin toplamına eşittir ve ayrıca her üç faz da bağımsız olduğundan varyans ta ayrı ayrı varyansların toplamına eşittir. Böylece,

$$E(X) = \sum_{i=1}^{3} \frac{1}{\mu_i} = \frac{1}{1} + \frac{1}{2} + \frac{3}{2},$$

$$Var(X) = \sum_{i=1}^{3} \frac{1}{{\mu_i}^2} = \frac{1}{1} + \frac{1}{4} = \frac{5}{4}$$

dır.

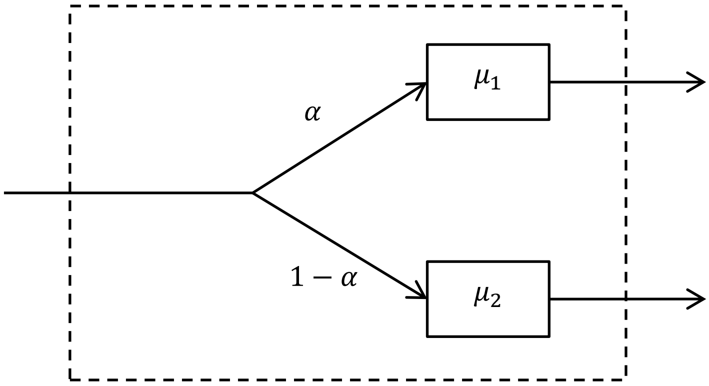
Böylece olasılık fonksiyonu aşağıdaki gibi elde edilir:

$$f_X(x) = -2(e^{-2x} - e^{-x})$$

$$= 2(e^{-x} - e^{-2x}), \qquad x \geq 0$$

Hiper-Üstel Dağılım

Hiper-üstel dağılımı aşağıda verilen şekil üzerinde anlatalım. Şekilde görüldüğü üzere, α_1 olasılıkla üst faz, $\alpha_2 = 1 - \alpha_1$ olasılıkla ile alttaki faz seçilir.



Bir hizmet tesisinin bu dağılım ile modellendiğini düşünelim. Bu taktirde hizmet tesisine gelen bir müşteri α_1 olasılıkla μ_1 parametreli üstel dağılıma sahip fazdan hizmetini alacak ve tesisten ayrılacak ya da α_2 olasılıkla μ_2 parametreli üstel dağılıma sahip fazdan hizmetini alacak ve tesisten ayrılacaktır. Aynı anda hizmet tesisinde birden fazla müşteri hizmet alamaz. Yani her iki faz aynı anda hizmet veremez.

Herhangi bir müşterinin hizmet süresini X tesadüfi değişkeni ile gösterelim. $X \sim H_2$ ile gösterilir ve X'in olasılık yoğunluk fonksiyonu

$$f_X(x) = \alpha_1\mu_1 e^{-\mu_1 x} + \alpha_2\mu_2 e^{-\mu_2 x}, \qquad x \geq 0$$

dır. Dağılım fonksiyonu ise aşağıdaki gibi bulunur,

$$F_X(x) = \alpha_1(1 - e^{-\mu_1 x}) + \alpha_2(1 - e^{-\mu_2 x}), \qquad x \geq 0.$$

Laplace dönüşümü,

$$L_X(s) = \alpha_1 \frac{\mu_1}{s + \mu_1} + \alpha_2 \frac{\mu_2}{s + \mu_2}$$

Beklenen Değer ve Varyans

$$E(X) = \frac{\alpha_1}{\mu_1} + \frac{\alpha_2}{\mu_2} \ \ ve\ E(X^2) = \frac{2\alpha_1}{{\mu_1}^2} + \frac{2\alpha_2}{{\mu_2}^2},$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2\alpha_1}{{\mu_1}^2} + \frac{2\alpha_2}{{\mu_2}^2} - \left[\frac{\alpha_1}{\mu_1} + \frac{\alpha_2}{\mu_2}\right]^2,$$

Örnek. X, parametreleri $\alpha_1 = 0{,}4, \mu_1 = 2\ ve\ \mu_2 = 1/2$ olan iki-fazlı bir tesadüfi değişken olsun. X'in beklenen değerini ve de standart sapmasını hesaplayalım.

Çözüm. İki-fazlı tesadüfi değişkenin ilgili formüllerinden direkt olarak,

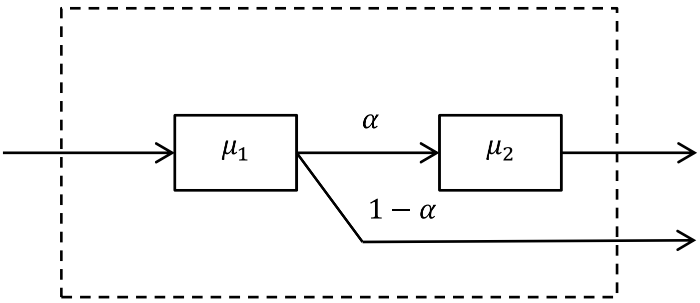
$$E(X) = \frac{0{,}4}{2} + \frac{0{,}6}{0{,}5} = 1{,}4\ , \qquad E(X^2) = \frac{0{,}8}{4} + \frac{1{,}2}{0{,}25} = 5,$$

$$\sigma_X = \sqrt{5 - (1{,}4)^2} = \sqrt{3{,}04} = 1{,}7436$$

olarak hesaplanır.

Coxian Dağılımı

Bu dağılımda fazlar birbirlerini takip eden tandem fazlar olarak ve herhangi bir fazda hizmet aldıktan sonra hizmet tesisinden ayrılmaya müsaade edecek şekilde düzenlenebilirler. Böylece, herhangi bir i fazında alınan hizmetten sonra müşteri α_i olasılıkla $(i+1)$ −inci faza devam edecek ya da $(1-\alpha_i)$ olasılıkla hizmet tesisinden ayrılacaktır. Bu durum iki fazlı Coxian dağılımı için aşağıda gösterilmiştir.



Coxian tesadüfi değişkeni $X{\sim}C_2$ ile gösterilir.

Laplace Dönüşümü

$$L_X(s) = (1-\alpha)\frac{\mu_1}{s+\mu_1} + \alpha\frac{\mu_1}{s+\mu_1}\frac{\mu_2}{s+\mu_2}.$$

Beklenen Değer ve Varyans

$$E(X) = \frac{1}{\mu_1} + \alpha\frac{1}{\mu_2} = \frac{\mu_2 + \alpha\mu_1}{\mu_1\mu_2}$$

$$E(X^2) = \frac{2}{{\mu_1}^2} + \frac{2\alpha}{{\mu_2}^2} + \frac{2\alpha}{\mu_1\mu_2}.$$

$$Var(X) = \left(\frac{2}{{\mu_1}^2} + \frac{2\alpha}{{\mu_2}^2} + \frac{2\alpha}{\mu_1\mu_2}\right) - \left(\frac{1}{\mu_1} + \frac{\alpha}{\mu_2}\right)^2$$

Örnek. Parametreleri $\mu_1 = 2, \mu_2 = 0{,}5 \; ve \; \alpha = 0{,}25$ olarak verilen bir Cox−2 tesadüfi değişkenin beklenen değerini, ikinci momentini ve varyansını bulunuz.

Çözüm.

$$E(X) = \frac{1}{\mu_1} + \frac{\alpha}{\mu_2} = \frac{1}{2} + \frac{1/4}{1/2} = 1,$$

$$E(X^2) = \frac{2}{{\mu_1}^2} + \frac{2\alpha}{{\mu_2}^2} + \frac{2\alpha}{\mu_1\mu_2} = \frac{2}{4} + \frac{1/2}{1/4} + \frac{1/2}{1} = 3,$$

$$Var(X) = E(X^2) - [E(X)]^2 = 3 - 1 = 2$$

Coxian dağılımıyla ilgili aşağıdaki şu sonuçlar elde edilir:

1. $\alpha = 0$ alınırsa üstel dağılıma döner.
2. $\alpha = 1$ alınırsa hipo-üstel dağılıma dönüşür.
3. $\alpha = 1, \; \mu_1 = \mu_2$ alınırsa Erlang dağılımına dönüşür.

Örnek: Makale Çalışması:

On optimization of a coxian queueing model with two phases

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Abstract: In this study we have obtained stochastic equation systems of a Coxian queueing model with two phases where arrival streamof this model is according to the exponential distribution with λ parameter. The service time of any customer at server i $(i = 1,2)$ is exponential with parameter μ_i. In addition we have obtained state probabilities of this queueing model at any given t moment.Furthermore performance measures of this queueing system are calculated. Various queueing systems are found for some values of α probability and service parameters: if $\alpha = 1$and $\mu_1 = \mu_2$taken then $M/E_2/1/\;0$ queueing model is obtained, for $\alpha = 1$it is shown that service time of a customer is according to hypoexponential, if $\alpha = 0$ is taken we have $M/\;M/1/\;0$ queueing system. Lately,an application of this queueing model is done. The optimal value of the mean customer number in the system is found. Finally, optimal ordering according to the loss probability is obtained by changing the service parameters .A numerical example is given on the subject

Keywords: Coxian Model, Differential Equations, Loss Probability, Limiting Distribution, Optimal Ordering

1. Introduction

The queueing systems with phase-type are one of the important part of the queueing theory. It is possible to construct phase-type distributions that are a mixture of hypoexponential and hyperexponential distributions to obtain increasingly complex representations. With the introduction of the mathematically expedient notion of “complex probabilities” and “complex rates,” [1] showed how any distribution having a rational Laplace transform could be represented by a sequence of exponential phases. The sequence of phase could be arranged one after the other in series formation, with the provision of permitting termination after the completion of any phase, explained in [2]. Another large subset of Coxian distributions is the phase-type distribution introduced by [3], which can be considered to be a natural probabilistic generalization of the Erlang, given in [4]. In [5], equilibrium state distributions were determined for queues with load-dependent Poisson arrivals and service time distributions representable by Cox's generalized method of stages. The solution was obtained by identifying a birth-death process that has the same equilibrium state distribution as the original queue. In [6], the optimal ordering of the tandem server with two stage is given. In [7], Optimal sample size is obtained depending on the probabilities of type 1 and type 2 errors in a queueing system with two channel in which service time has Coxian distribution and arrivals are Poisson distribution. In this paper we have obtained steady state equation systems of a Coxian queueing model with two phases where the arrivals to this model are according to the exponential distribution with λ parameter . We have obtained state probabilities of the system with respect to α at any given t moment. The optimal values of loss probability and the mean customer number in the system are found. Furthermore, loss probability by changing the service parameters are obtained for optimal ordering of phases. In this system if $\alpha = 1$ and $\mu_1 = \mu_2$ taken then $M/E_2/1/0$ queueing model is obtained. Also for $\alpha = 1$ it is shown that service time of a customer is according to hypoexponential. If $\alpha = 0$ is taken we have $M/M/1/0$ queueing system. In addition an application of this queueing model is given

2. Stochastic Model

We have obtained stochastic equation systems of a Coxian queueing model with two phases where the incomes to this model are according to the exponential distribution with λ parameter . The service time of any customer at phases i ($i = 1,2$) is exponential with parameter μ_i. First phase and second phase can be empty or busy but in this system it is not allowed to be two customers in a phase at the same time. Let ξ_t be the state of first phase and η_t be the state of second phase at any t moment. We have obtained the state probabilities with respect to α at any given t moment. Limit probabilities, differential and difference equations of this model given later.

2.1. LimitDistribution

Here $\{(\xi_t, \eta_t), t \geq 0\}$ is a 2 - dimensional Markov chain with continuous parameter and state space
Ω= $\{(0,0), (0,1), (1,0))\}$.

$$P_{ij}(t) = \text{Prob}\{\xi_t = i, \eta_t = j\}. \forall (i,j) \in \Omega \quad (1)$$

Kolmogorov differential equation for these probabilities is obtained. The transient probabilities of the process $\{(\xi_t, \eta_t), t \geq 0\}$ is found for $(t, t+h)$, namely

$$P_{00}(t+h) = \big(1 - \lambda h + o(h)\big)P_{00}(t) + (1-\alpha)\big(\mu_1 h + o(h)\big)P_{10}(t) + \big(\mu_2 h + 0(h)\big)P_{01}(t) + o(h) \quad (2)$$

$$P_{01}(t+h) = \big(1 - \mu_2 h + o(h)\big)P_{01}(t) + \alpha\big(\mu_1 h + o(h)\big)P_{10}(t) + o(h) \quad (3)$$

$$P_{10}(t+h) = \big(1 - \mu_1 h + o(h)\big)P_{10}(t) + \big(\lambda h + o(h)\big)P_{00}(t) + o(h) \quad (4)$$

We write (2), (3) and (4) equations as follows as $h \longrightarrow 0$,

$$P'_{00}(t) = -\lambda P_{00}(t) + (1-\alpha)\mu_1 P_{10}(t) + \mu_2 P_{01}(t) \quad (5)$$

$$P'_{01}(t) = -\mu_2 P_{01}(t) + \alpha\mu_1 P_{10}(t) \quad (6)$$

$$P'_{10}(t) = -\mu_1 P_{10}(t) + \lambda P_{00}(t) \quad (7)$$

It is supposed that limiting distribution of $P_{ij}(t)$ are exist as followings:

$$\lim_{t\to\infty} P_{ij}(t) = \pi_{ij} \quad (8)$$

$$\lim_{t\to\infty} P'_{ij}(t) = 0 \quad (9)$$

Limiting distribution has been widely given in [8].

Steady- state equations for $\{(\xi_t, \eta_t), t \geq 0\}$ are obtained as following:

$$0 = -\lambda\pi_{00} + (1-\alpha)\mu_1\pi_{10} + \mu_2\pi_{01} \quad (10)$$

$$0 = -\mu_2\pi_{01} + \alpha\mu_1\pi_{10} \quad (11)$$

$$0 = -\mu_1\pi_{10} + \lambda\pi_{00} \quad (12)$$

$$\sum\sum_{(i,j)\epsilon\Omega} \pi_{ij} = 1 \quad (13)$$

We define $\rho_1 = \lambda/\mu_1$ and $\rho_2 = \lambda/\mu_2$. If the equations (10), (11) and (12) are solved, the following transient probabilities are obtained:

$$\pi_{01} = \alpha\rho_2\pi_{00} \quad (14)$$

$$\pi_{10} = \rho_1\pi_{00} \quad (15)$$

If the obtained transient probabilities and π_{00} are put in the equation (13),

$$\pi_{00}(1 + \rho_1 + \alpha\rho_2) = 1 \quad (16)$$

$$\pi_{00} = \frac{1}{1 + \rho_1 + \alpha\rho_2} \quad (17)$$

is found. π_{01} and π_{10} probabilities are calculated if the equation (17) is put in (14) and (15).

2.2. Probability Function

We define $\rho_1 = \lambda/\mu_1$ and $\rho_2 = \lambda/\mu_2$. If we shall solve (10), (11) and (12) equations under condition (13), we obtain the following two dimension probability function:

$$\pi_{ij} = \begin{cases} \dfrac{1}{1+\rho_1+\rho_2\alpha}, & (i,j) = (0,0) \\ \dfrac{\rho_2\alpha}{1+\rho_1+\rho_2\alpha}, & (i,j) = (0,1) \\ \dfrac{\rho_1}{1+\rho_1+\rho_2\alpha}, & (i,j) = (1,0) \\ 0, & otherwise. \end{cases} \quad (18)$$

Let π_0 be the probability of being no customer in system and π_1 to be the probability of being one customer in the system. Where,

$$\pi_0 = \pi_{00}, \quad \pi_1 = \pi_{01} + \pi_{10} \quad (19)$$

3. Obtaining the Measures of Performance

3.1. Coxian Queue Using z-Transform

The z-transform of a sequence π_k, $k = 0,1, \ldots$, is defined as

$$P(z) = \sum_{k=0}^{\infty} \pi_k z^k, \tag{20}$$

Where z is complex variable and is such that $\mathrm{P}(z)$ is analytic i.e., $\sum_{k=0}^{\infty} \pi_k z^k < \infty$, in[2].

$$P(z) = \pi_0 + \pi_1 z$$
$$= \frac{1 + (\rho_1 + \rho_2\alpha)z}{1 + \rho_1 + \rho_2\alpha} \tag{21}$$

Let N be the random variable that describes the number of customers in the system.The mean number of costumers:

$$E[N] = \frac{d}{dz}P(z)\Big|_{z=1} = \frac{\rho_1 + \rho_2\alpha}{1 + \rho_1 + \rho_2\alpha}\cdot \tag{22}$$

Second and higher moments may be computed from correspondingly higher derivatives. The k^{th} derivative of the z −transform evaluated at $z = 1$ gives the k^{th} factorial moment:

$$\lim_{z\to 1} P^{(k)}(z) = E[N(N-1)\ldots(N-k+1)]. \tag{23}$$

Thus,

$$Var(N) = E([N(N+1)] + E[N] - E^2[N])$$
$$= \frac{\rho_1 + \rho_2\alpha}{(1 + \rho_1 + \rho_2\alpha)^2}\cdot \tag{24}$$

3.2. Coxian Queue Using Laplace Transform

Let W be the random variable that describes waiting time of customers in the system. Laplace transform of W

$$\mathcal{L}_W(s) = \frac{(1-\alpha)\mu_1}{\mu_1 + s} + \frac{\alpha\mu_1}{\mu_1 + s}\frac{\mu_2}{\mu_2 + s}\cdot \tag{25}$$

Mean waiting time in system of a customerfor *Cox*(2) is found by formula (20).

$$E[W] = \frac{\mu_2 + \alpha\mu_1}{\mu_1\mu_2}\cdot \tag{26}$$

By using the formula (19) are also obtained other moments. For example,

$$Var[W] = \frac{\mu_2^2 + \alpha\mu_1^2(2-\alpha)}{\mu_1^2\mu_2^2}\cdot \tag{27}$$

3.3. The Optimization of Measures of Performance

Theorem 1.$E[N]$ is maximum for $\alpha = 1$.
Proof. . We can rewrite equation (22) as following:

$$\frac{1}{E[N]} = 1 + \frac{1}{\rho_1 + \rho_2\alpha}\cdot \qquad 0 \le \alpha \le 1 \tag{28}$$

The minimum value of $1/E[N]$ is the maximum value of $E[N]$. $1/E[N]$is minimum for $\alpha = 1$. In other words,the mean number of costumers is maximum with probability $\alpha = 1$.

$$\max_{0\le\alpha\le 1} E[N] = \frac{\rho_1 + \rho_2}{1 + \rho_1 + \rho_2}\cdot \tag{29}$$

Loss probability
Let Π_{loss} be the loss probability of customer in the system. In this regards, since there is no queue in the system,loss probability is calculated as following

$$\boldsymbol{\Pi_{loss} = \pi_{01} + \pi_{10} = 1 - \pi_{00}} \tag{30}$$

$$\Pi_{loss} = 1 - \frac{1}{1 + \rho_1 + \rho_2\alpha} \tag{30$'$}$$

3.4. Optimal Order of Servers

Considering$0 \le \alpha \le 1$, letμ_1 denotes the service parameter of the first phase and the second phase andμ_2 denotes theservice parameter of second phase. Indication of loss probabilities and measure of performance for $\mu_1 \ge \mu_2$ are illustrated with **(1)**. Similarly indication of loss probabilities and measure of performance for $\mu_1 \le \mu_2$ are illustrated with **(2)**. arrival

By using the formula (24), we write $\Pi_{loss}^{(1)}$ and $\Pi_{loss}^{(2)}$ as follows:

$$\Pi_{loss}^{(1)} = \frac{\rho_1 + \rho_2\alpha}{1 + \rho_1 + \rho_2\alpha} \tag{31}$$

$$\Pi_{loss}^{(2)} = \frac{\rho_2 + \rho_1\alpha}{1 + \rho_2 + \rho_1\alpha} \tag{32}$$

The following theorem is given for optimal ordering.

Theorem 2. In this system, if the parameter of the first phase greater than the parameter of the second phase,

$$\Pi_{loss}^{(1)} \le \Pi_{loss}^{(2)}. \tag{33}$$

Proof.$\mu_2 \le \mu_1$ (34)
This inequalityisrewritten as following

$$\frac{1}{\mu_1} \le \frac{1}{\mu_2} \tag{35}$$

$$\rho_1 \le \rho_2 \tag{36}$$

$$\rho_1(1-\alpha) \le \rho_2(1-\alpha) \tag{37}$$

$$\rho_1 + \rho_2\,\alpha \le \rho_2 + \rho_1\alpha \tag{38}$$

$$\frac{1}{\rho_2 + \rho_1\alpha} \le \frac{1}{\rho_1 + \rho_2\alpha} \tag{39}$$

$$1+\frac{1}{\rho_2+\rho_1\alpha} \leq 1+\frac{1}{\rho_1+\rho_2\alpha} \qquad (40)$$

$$\left[\frac{1+\rho_1+\rho_2\,\alpha}{\rho_1+\rho_2\,\alpha}\right]^{-1} \leq \left[\frac{1+\rho_2+\rho_1\,\alpha}{\rho_2+\rho_1\,\alpha}\right]^{-1} \qquad (41)$$

3.5. Queueing models derived from $M/Cox(2)/1/\,0$

The $M/Cox(2)/1/\ 0$ stochastic model is illustrated in figure1.

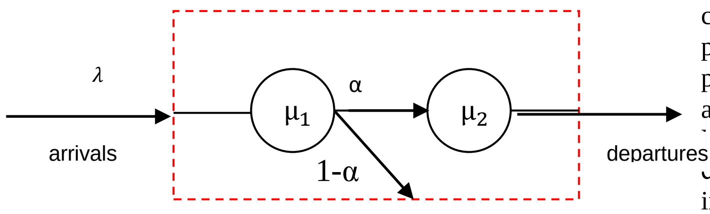

Figure 1.Coxianqueueing model with two phases.

In $M/Cox(2)\ /1/\ 0$ stochastic model if $\alpha = 1$ and $\mu_1 = \mu_2$ chosen then $M\,/\,E_2\,/\,1\,/\,0$ queueing system is obtained and this system is illustrated in figure 2.

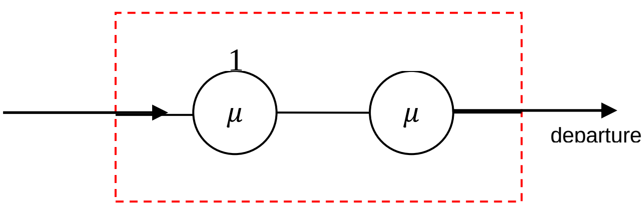

Figure 2. $M\,/\,E_2\,/\,1\,/\,0$ queueing model.

In $M/Cox(2)/1/\,0$ queueing model if $\alpha = 1$ and $\mu_1 \neq \mu_2$ then $M\,/Hyp(\,2)\,/\,1\,/\,0$ is obtained and this queueing model is illustrated in figure 3

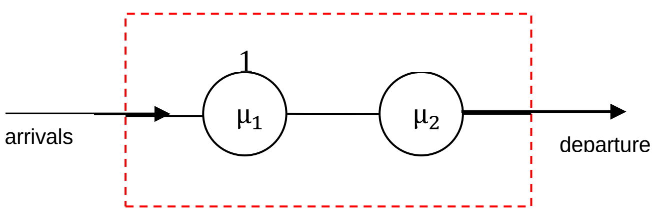

Figure 3. The. $M\,/Hyp(\,2)\,/\,1\,/\,0$ queuing model.

Finally the $M/Cox(2)/1/\,0$ model turns into the system $M/M/1/0$ for $\alpha = 0$ and this system is illustrated in figure4.

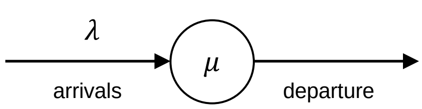

Figure 4. The $M/M/1/0$ queueing model.

As illustrated above, 3 different queueing models derived from $M/Cox(2)/1/\,0$ system. There is no waiting at all these queueing models.

4. Numerical Example

In an interview consists of two juries, the arrivals of the candidates are according to Poisson stream with the parameter $\lambda = 2{,}3$, no waiting allowed and the service parameters are $\mu_1 = 5{,}2$ and $\mu_2 = 3{,}2$ respectively according to Coxian distribution. For different values of α, loss probabilities and measure of performances are calculated and given in Table 1. When the juries' places in interview are changed (changing the parameters places) loss probabilities and measureof performances are calculated as well in Table 2. Assuming that $\mu_1 = \mu_2 = 4{,}2$, loss probabilities and measure of performances are obtained in Table 3.

Table 1.*For* $\lambda = 2.3,\ \mu_1 = 5{,}2\ ;\ \mu_2 = 3{,}2.$

α	$\Pi^{(1)}_{loss}$	$Max\{E(N)\}$	$Var^{(1)}(N)$	$E^{(1)}(W)$	$Var^{(1)}(W)$
0,0	0,278	0,513	0,201	0,172	0,029
0,4	0,395	0,513	0,239	0,291	0,087
0,6	0,440	0,513	0,246	0,351	0,104
1,0	0,513	0,513	0,250	0,470	0,119

Table 2.*For* $\lambda = 2.3,\ \mu_1 = 3{,}2\ ;\ \mu_2 = 5{,}2.$

α	$\Pi^{(2)}_{loss}$	$Max\{E(N)\}$	$Var^{(2)}\,(N)$	$E^{(2)}(W)$	$Var^{(2)}(W)$
0,0	0,401	0,513	0,240	0,299	0,089
0,4	0,451	0,513	0,248	0,367	0,108
0,6	0,474	0,513	0,249	0,402	0,114
1,0	0,513	0,513	0,250	0,470	0,119

Table 3.*For* $\lambda = 2.3,\ \mu_1 = \mu_2 = 4{,}2.$

α	Π_{loss}	$Max\{E(N)\}$	$Var(N)$	$E(W)$	$Var(W)$
0,0	0.353	0.522	0.228	0.238	0.056
0,4	0.433	0.522	0.245	0.333	0.092
0,6	0.466	0.522	0.248	0.380	0.104
1,0	0.522	0.522	0.249	0.476	0.113

5.Conclusion

By analyzing this system, limit probabilities and probability mass function are obtained using generating function. The mean number of customer in the system and the loss probability of any customer are calculated. The optimization of the mean number of customer in the system is proved by Theorem 1: For both ordering, the optimal mean customer number in the system which is $Max\{E(N)\}$ is found to be the same value. The optimal ordering of phases for service parameters is found. It is observed that the first order is optimal according to measures of performances. The variance and Laplace transform of mean waiting time of a customer in system are given: As the mean service time is minimum for $\alpha = 0$, variance of service time is minimum for $\alpha = 1$. A numerical example is given on the subject. Furthermore it is shown that this analyzed system turns into different queueing systems when concerning the probability of customer's leaving the system and service parameters are same or different. The system M / *Cox(2)* / 1 / 0 for $\alpha = 1$ and $\mu_1 = \mu_2$ turns into the system M / E_2 / 0 : This situation is numerically shown. The system M /*Cox(2)*/ 1/ 0 for $\alpha = 1$ and $\mu_1 \neq \mu_2$ turns into the system M /Hyp(2)/ 1, in other words service time has hypo-exponential distribution and this situation is numerically shown. The system M / *Cox(2)*/ 1/0 for $\alpha = 0$turns into the system M / M / 1/ 0: Thus situations are numerically shown. For $\alpha < 1$, depending on the order it is clear that mean waiting time, the variance of waiting time and the variance of customer number in the system is less in first order. For later studies, increasing the number of phases may be advisable.

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