

DENKLEM VE DENKLEM SİSTEMLERİNİN ÇÖZÜMÜ (Devam)

• **Cholesky Yöntemi** : Bu yöntemde $A X = b$ şeklindeki denklem takımında A matrisinin L alt üçgen matrisi ile köşegeni 1'e eşit olan U üçgen matrisinin çarpımı şeklinde oluştuğu düşünülür ve L ve U matrisleri hesaplanır.

$\left. \begin{array}{l} AX = b \\ A = L.U \end{array} \right\} \begin{array}{l} L.U.X = b \\ L.Y = b \end{array}$ eşitliklerinden Y bilinmeyeni bulunur. Daha sonra $U.X = Y$ eşitliğinden de X bilinmeyenleri bulunur.

L ve U matrislerinin elemanları;

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{lll} l_{11} = a_{11} & u_{12} = a_{12}/l_{11} & u_{13} = a_{13}/l_{11} \\ l_{21} = a_{21} & l_{22} = a_{22} - l_{21}u_{12} & u_{23} = (a_{23} - l_{21}u_{13})/l_{22} \\ l_{31} = a_{31} & l_{32} = a_{32} - l_{31}u_{12} & l_{33} = a_{33} - l_{32}u_{23} - l_{31}u_{13} \end{array}$$

şeklinde bulunur.

$$\mathbf{\ddot{O}rnek :} \left. \begin{array}{l} x + y + z = 6 \\ 2x + 3y + z = 1 \\ x - y + z = 3 \end{array} \right\} \text{denklem sisteminin k\u00f6klerini Cholesky Y\u00f6ntemine g\u00f6re bulunuz.}$$

$$\left. \begin{array}{l} x + y + z = 6 \\ 2x + 3y + z = 1 \\ x - y + z = 3 \end{array} \right\} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix}$$

$$\text{L matrisi} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2 & -2 \end{bmatrix}, \text{U matrisi} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{L.Y} = \text{b} \Rightarrow \text{Y} = \begin{bmatrix} 6 \\ -11 \\ 12.5 \end{bmatrix}, \text{U.X} = \text{Y} \Rightarrow \text{X} = \begin{bmatrix} -8 \\ 1.5 \\ 12.5 \end{bmatrix}$$

3.3.1.2. Katsayılar Matrisi Simetrik Olan Denklem Sistemlerinin Çözümü

- **Modernleştirilmiş Gauss Algoritması Yöntemi:**

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_3 \\ a_{12}x_1 + a_{22}x_2 + a_{23}x_3 = b_3 \\ a_{13}x_1 + a_{23}x_2 + a_{33}x_3 = b_3 \end{array} \right\} \begin{array}{l} S_1 = a_{11} + a_{12} + a_{13} \\ S_2 = a_{12} + a_{22} + a_{23} \\ S_3 = a_{13} + a_{23} + a_{33} \end{array} \text{ eşitlikleri tablo ile}$$

X₁	X₂	X₃	-b	S
a ₁₁	a ₁₂	a ₁₃	-b ₁	S ₁
-1	-a ₁₂ /a ₁₁	-a ₁₃ /a ₁₁	b ₁ /a ₁₁	-S ₁ /a ₁₁
	a ₂₂	a ₂₃	-b ₂	S ₂
	a _{22.1} = a ₂₂ - a ₁₂ ·a ₁₂ /a ₁₁	a _{23.1} = a ₂₃ - a ₁₂ ·a ₁₃ /a ₁₁	-b _{2.1} = -b ₂ + a ₁₂ ·b ₁ /a ₁₁	S _{2.1} = S ₂ - a ₁₂ ·S ₁ /a ₁₁
	-1	-a _{23.1} /a _{22.1}	b _{2.1} /a _{22.1}	-S _{2.1} /a _{22.1}
		a ₃₃	-b ₃	S ₃
		a _{33.2} = a ₃₃ - a ₁₃ ·a ₁₃ /a ₁₁ - a _{23.1} ·a _{23.1} /a _{22.1}	-b _{3.2} = -b ₃ + a ₁₃ ·b ₁ /a ₁₁ + a _{23.1} ·b _{2.1} /a _{22.1}	S _{3.2} = S ₃ - a ₁₃ ·S ₁ /a ₁₁ - a _{23.1} ·S _{2.1} /a _{22.1}
		-1	b _{3.2} /a _{33.2}	-S _{3.2} /a _{33.2}

$$x_3 = \frac{b_{3.2}}{a_{33.2}}, x_2 = -\frac{a_{23.1}}{a_{22.1}} \cdot x_3 + \frac{b_{2.1}}{a_{22.1}}, x_1 = -\frac{a_{12}}{a_{11}} x_2 - \frac{a_{13.1}}{a_{11}} \cdot x_3 + \frac{b_1}{a_{11}}$$

Toplam sütununun kontrolleri yapılmalıdır.

Ters Matrisin hesabı:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}^{-1} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} (-(-1)/a_{11}) - (a_{12}/a_{11}).q_{12} - (a_{13}/a_{11}).q_{13} & -(a_{12}/a_{11}).q_{22} - (a_{13}/a_{11}).q_{23} & -(a_{12}/a_{11}).q_{23} - (a_{13}/a_{11}).q_{13} \\ q_{12} & -(-1)/a_{22.1} - (a_{23.1}/a_{22.1}).q_{23} & -(a_{23.1}/a_{22.1}).q_{33} \\ q_{13} & q_{23} & -(-1)/a_{33.2} \end{bmatrix}$$

Örnek :

$$16x_1 + 8x_2 + 12x_3 - 4x_4 = 44$$

$$8x_1 + 5x_2 + 11x_3 - 4x_4 = 30$$

$$12x_1 + 11x_2 + 70x_3 - 31x_4 = 145$$

$$-4x_1 - 4x_2 - 31x_3 + 63x_4 = -161$$

X₁	X₂	X₃	X₄	-b	S
16	8	12	-4	-44	-12
-1	-8/16=-0.5	-12/16=-0.75	4/16=0.25	44/16=2.75	12/16=0.75
	5	11	-4	-30	-10
	5+8*(-0.5)=1	11+8*(-0.75)=5	-4+(8)*0.25=-2	-30+(8)*2.75=-8	-10+(8)*0.75=-4
	-1	-5/1=-5	2/1=2	8/1=8	4/1=4
		70	-31	-145	-83
		70+12*(-0.75)+5*5=36	-31+12*0.25+5*2=-18	-145+33+40=-72	-83+9+20=-54
		-1	18/36=0.5	72/36=2	54/36=1.5
			63	161	185
			63-1-4-9=49	161-11-16-36=98	185-3-8-27=147
			-1	98/-49=2	147/-49=-3

$$-x_4 - 2 = 0 \Rightarrow x_4 = -2$$

$$-x_3 + 0.5x_4 + 2 = 0 \Rightarrow x_3 = 1$$

$$-x_2 - 5x_3 + 2x_4 + 8 = 0 \Rightarrow x_2 = -1$$

$$-x_1 - 0.5x_2 - 0.75x_3 + 0.25x_4 + 2.75 = 0 \Rightarrow x_1 = 2$$

Ters Matris Hesabı

$$q_{44} = -\frac{-1}{49} = 0.0204; \quad q_{34} = (0.5) * 0.0204 = 0.0102; \quad q_{24} = -5 * 0.0102 + (2) * 0.0204 = -0.0102 \quad ;$$
$$q_{14} = (-0.5) * (-0.0102) + (-0.75) * 0.0102 + 0.25 * 0.0204 = 0.0026$$

$$q_{33} = -\frac{-1}{36} + (0.5) * 0.0102 = 0.0329; \quad q_{23} = (-5) * 0.0329 + 2 * 0.0102 = -0.1441; \quad q_{13} = (-0.5) * -0.1441 + (-0.75) * 0.0329 + 0.25 * 0.0102 = 0.0499$$

$$q_{22} = -\frac{-1}{1} + (-5) * (-0.1441) + 2 * (-0.01022) = 1.7000; \quad q_{12} = (-0.5) * 1.7000 + (-0.75) * (-0.1441) + 0.25 * (-0.01022) = -0.7445$$

$$q_{11} = -\frac{-1}{16} + (-0.5) * (-0.7445) + (-0.75) * (0.0499) + 0.25 * 0.0026 = 0.3980$$

$$Q = \begin{bmatrix} 0.3980 & -0.7445 & 0.0499 & 0.0026 \\ -0.7445 & 1.7000 & -0.1441 & -0.0102 \\ 0.0499 & -0.1441 & 0.0329 & 0.0102 \\ 0.0026 & -0.0102 & 0.0102 & 0.0204 \end{bmatrix}$$

$$4x_1 - 2x_2 + 6x_3 - 12x_4 = 4$$

Örnek : $-2x_1 + 3x_2 + x_3 + 2x_4 = -4$

$$6x_1 + x_2 + 20x_3 - 32x_4 = 5$$

$$-12x_1 + 2x_2 - 32x_3 + 60x_4 = -10$$

X₁	X₂	X₃	X₄	-b	S
4	-2	6	-12	-4	-8
-1	0.5	-1.5	3	1	2
	3	1	2	4	8
	2	4	-4	2	4
	-1	-2	2	-1	-2
		20	-32	-5	-10
		3	-6	-3	-6
		-1	2	1	2
			60	10	28
			4	-4	0
			-1	1	0

$$x_4=1, x_3=3, x_2=-5, x_1=-3$$

$$Q = \begin{bmatrix} 2.7981 & 2.4175 & -1.3329 & -0.2500 \\ 2.4175 & 2.8340 & -1.6667 & -0.5000 \\ -1.3329 & -1.6667 & 1.3333 & 0.5000 \\ -0.2500 & -0.5000 & 0.5000 & 0.2500 \end{bmatrix} \text{ üst üçgen matris } \begin{bmatrix} 1 & -0.5 & 1.5 & -3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

• **Cholesky (Karekök) Yöntemi** : Bu yöntemde simetrik A matrisinin Cholesky indirgemesiyle $A = C^T C$ şeklindeki C üst üçgen matrisi hesaplanır.

$$\left. \begin{array}{l} AX = b \\ A = C^T C \\ b = C^T d \end{array} \right\} \text{ ise denklem } C^T C X = C^T d \Rightarrow C X = d \text{ şekline gelir.}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & \dots & 0 \\ c_{12} & c_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ c_{1n} & c_{2n} & \dots & c_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ 0 & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c_{nn} \end{bmatrix} \text{ eşitliğinden önce C matrisi}$$

$$\begin{bmatrix} c_{11} & 0 & \dots & 0 \\ c_{12} & c_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ c_{1n} & c_{2n} & \dots & c_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \dots \\ d_n \end{bmatrix} \text{ eşitliğinden ise d vektörü hesaplanır.}$$

C matrisi ve d vektörü;

❖ 1. satır elemanları; $c_{11} = \sqrt{a_{11}}$, $c_{1j} = a_{1j}/c_{11}$ $j = 2, 3, \dots, n$

❖ Köşegen elemanları; $c_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} c_{ki}^2}$, $i = 2, 3, \dots, n$

❖ Diğer elemanları; $c_{ij} = \frac{1}{c_{ii}} \left(a_{ij} - \sum_{k=1}^{i-1} c_{ki} c_{kj} \right)$, $i = 2, 3, \dots, n-1$ $j = i+1, i+2, \dots, n$

❖ d vektörü elemanları $d_1 = b_1/c_{11}$, $d_i = \frac{1}{c_{ii}} \left(b_i - \sum_{k=1}^{i-1} c_{ki} d_k \right)$ $i = 2, 3, \dots, n$

eşitlikleriyle hesaplanır. $CX = d$ eşitliğinden X bilinmeyenleri bulunur.

X₁	X₂	X₃	-b	S
a₁₁	a₁₂	a₁₃	-b₁	S₁
$\sqrt{a_{11}}$	$a_{12}/\sqrt{a_{11}}$	$a_{13}/\sqrt{a_{11}}$	$-b_1/\sqrt{a_{11}}$	$S_1/\sqrt{a_{11}}$
	a₂₂	a₂₃	-b₂	S₂
	$a_{22.1} = a_{22} - a_{12}/\sqrt{a_{11}} \cdot a_{12}/\sqrt{a_{11}}$	$a_{23.1} = a_{23} - a_{12}/\sqrt{a_{11}} \cdot a_{13}/\sqrt{a_{11}}$	$b_{2.1} = -b_2 + a_{12}/\sqrt{a_{11}} \cdot b_1/\sqrt{a_{11}}$	$S_{2.1} = S_2 - a_{12}/\sqrt{a_{11}} \cdot S_1/a_{11}$
	$\sqrt{a_{22.1}}$	$a_{23.1}/\sqrt{a_{22.1}}$	$b_{2.1}/\sqrt{a_{22.1}}$	$S_{2.1}/\sqrt{a_{22.1}}$
		a₃₃	-b₃	S₃
		$a_{33.2} = a_{33} - a_{13}/\sqrt{a_{11}} \cdot a_{13}/\sqrt{a_{11}}$ $- a_{23.1}/\sqrt{a_{22.1}} \cdot a_{23.1}/\sqrt{a_{22.1}}$	$b_{3.2} = -b_3 + a_{13}/\sqrt{a_{11}} \cdot b_1/\sqrt{a_{11}}$ $- a_{23.1}/\sqrt{a_{22.1}} \cdot b_{2.1}/\sqrt{a_{22.1}}$	$S_{3.2} = S_3 - a_{13}/\sqrt{a_{11}} \cdot S_1/\sqrt{a_{11}}$ $- a_{23.1}/\sqrt{a_{22.1}} \cdot S_{2.1}/\sqrt{a_{22.1}}$
		$\sqrt{a_{33.2}}$	$b_{3.2}/a_{33.2}$	$S_{3.2}/\sqrt{a_{33.2}}$

Örnek :

$$16x_1 + 8x_2 + 12x_3 - 4x_4 = 44$$

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$$-4x_1 - 4x_2 - 31x_3 + 63x_4 = -161$$

X₁	X₂	X₃	X₄	-b	S
16 $\sqrt{16} = 4$	8 $8/\sqrt{16} = 2$	12 $12/\sqrt{16} = 3$	-4 $-4/\sqrt{16} = -1$	-44 $-44/\sqrt{16} = -11$	-12 $-12/\sqrt{16} = -3$
	5 $5 - 2 \cdot 2 = 1$ 1	11 $11 - 2 \cdot 3 = 5$ 5	-4 $-4 - 2 \cdot (-1) = -2$ -2	-30 $-30 - 2 \cdot (-11) = -8$ -8	-10 $-10 - 2 \cdot (-3) = -4$ -4
		70 $70 - 9 - 25 = 36$ 6	-31 $-31 + 3 + 10 = -18$ $-18/6 = -3$	-145 $-145 + 33 + 40 = -72$ $-72/6 = -12$	-83 $-83 + 9 + 20 = -54$ $-54/6 = -9$
			63 $63 - 1 - 4 - 9 = 49$ 7	161 $161 - 11 - 16 - 36 = 98$ $98/7 = 14$	185 $185 - 3 - 8 - 27 = 147$ $147/7 = 21$

$$7x_4 + 14 = 0 \Rightarrow x_4 = -2$$

$$6x_3 - 3x_4 - 12 = 0 \Rightarrow x_3 = 1$$

$$x_2 + 5x_3 - 2x_4 - 8 = 0 \Rightarrow x_2 = -1$$

$$4x_1 + 2x_2 + 3x_3 - x_4 - 11 = 0 \Rightarrow x_1 = 2$$

Örnek :

$$4x_1 - 2x_2 + 6x_3 - 12x_4 = 4$$

$$-2x_1 + 3x_2 + x_3 + 2x_4 = -4$$

$$6x_1 + x_2 + 20x_3 - 32x_4 = 5$$

$$-12x_1 + 2x_2 - 32x_3 + 60x_4 = -10$$

X₁	X₂	X₃	X₄	-b	S
4	-2	6	-12	-4	-8
2	-1	3	-6	-2	-4
	3	1	2	4	8
	2	4	-4	2	4
	1.414	2.829	-2.829	1.414	2.829
		20	-32	-5	-10
		2.934	-5.997	-3.000	-6.003
		1.713	-3.500	-1.751	-3.504
			60	10	28
			3.747	-4.128	-0.261
			1.936	-2.132	0.135

$$x_4=1.101, x_3=3.272, x_2=-5.343, x_1=-3.276$$