CHAPTER 13

LARGE DEFLECTIONS OF PLATES

96. Bending of Circular Plates by Moments Uniformly Distributed along the Edge. In the previous discussion of pure bending of circular plates it was shown (see page 47) that the strain of the middle plane of the plate can be neglected in cases in which the deflections are small as compared with the thickness of the plate. In cases in which the deflections are no longer small in comparison with the thickness of the plate but are still small as compared with the other dimensions, the analysis of the problem must be extended to include the strain of the middle plane of the plate.\(^1\)

We shall assume that a circular plate is bent by moments \(M_0\) uniformly distributed along the edge of the plate (Fig. 200a). Since the deflection surface in such a case is symmetrical with respect to the center \(O\), the displacement of a point in the middle plane of the plate can be resolved into two components: a component \(u\) in the radial direction and a component \(w\) perpendicular to the plane of the plate. Proceeding as previously indicated in Fig. 196 (page 384), we conclude that the strain in the radial direction is\(^2\)

\[
\varepsilon_r = \frac{du}{dr} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 \quad (a)
\]

The strain in the tangential direction is evidently

\[
\varepsilon_t = \frac{u}{r} \quad (b)
\]

Denoting the corresponding tensile forces per unit length by \(N_r\) and

\(^1\) This problem has been discussed by S. Timoshenko; see Mem. Inst. Ways Commun., vol. 89, St. Petersburg, 1915.

\(^2\) In the case of very large deflections we have

\[
\varepsilon_r = \frac{du}{dr} + \frac{1}{2} \left[ \left( \frac{du}{dr} \right)^2 + \left( \frac{dw}{dr} \right)^2 \right]
\]

and applying Hooke's law, we obtain

\[
N_r = \frac{Eh}{1 - \nu^2} (\varepsilon_r + \nu \varepsilon_t) = \frac{Eh}{1 - \nu^2} \left[ \frac{du}{dr} + \frac{1}{2} \left( \frac{d\omega}{dr} \right)^2 + \nu \frac{u}{r} \right]
\]

\[
N_t = \frac{Eh}{1 - \nu^2} (\varepsilon_t + \nu \varepsilon_r) = \frac{Eh}{1 - \nu^2} \left[ \frac{u}{r} + \nu \frac{du}{dr} + \nu \left( \frac{d\omega}{dr} \right)^2 \right]
\]

(c)

These forces must be taken into consideration in deriving equations of equilibrium for an element of the plate such as that shown in Fig. 200b and c. Taking the sum of the projections in the radial direction of all the forces acting on the element, we obtain

\[
r \frac{dN_r}{dr} \, dr \, d\theta + N_r \, dr \, d\theta - N_t \, dr \, d\theta = 0
\]

from which

\[
N_r - N_t + r \frac{dN_r}{dr} = 0
\]

(d)
The second equation of equilibrium of the element is obtained by taking moments of all the forces with respect to an axis perpendicular to the radius in the same manner as in the derivation of Eq. (55) (page 53). In this way we obtain

\[ Q_r = -D \left( \frac{d^2w}{dr^2} + \frac{1}{r} \frac{d^2w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} \right) \]  \hfill (e)

The magnitude of the shearing force \( Q_r \) is obtained by considering the equilibrium of the inner circular portion of the plate of radius \( r \) (Fig. 200a). Such a consideration gives the relation

\[ Q_r = -N_r \frac{dw}{dr} \]  \hfill (f)

Substituting this expression for shearing force in Eq. (e) and using expressions (c) for \( N_r \) and \( N_t \) we can represent the equations of equilibrium (d) and (e) in the following form:

\[
\begin{align*}
\frac{d^2u}{dr^2} &= -\frac{1}{r} \frac{du}{dr} + \frac{u}{r^2} - \frac{1}{2r} \left( \frac{dw}{dr} \right)^2 - \frac{dw}{dr} \frac{d^2w}{dr^2} \\
\frac{d^3w}{dr^3} &= -\frac{1}{r} \frac{d^2w}{dr^2} + \frac{1}{r^2} \frac{dw}{dr} + \frac{12}{h^2} \frac{dw}{dr} + \frac{\nu u}{h^2} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2
\end{align*}
\]  \hfill (231)

These two nonlinear equations can be integrated numerically by starting from the center of the plate and advancing by small increments in the radial direction. For a circular element of a small radius \( c \) at the center, we assume a certain radial strain

\[ \epsilon_0 = \left( \frac{du}{dr} \right)_{r=0} \]

and a certain uniform curvature

\[ \frac{1}{\rho_0} = - \left( \frac{d^2w}{dr^2} \right)_{r=0} \]

With these values of radial strain and curvature at the center, the values of the radial displacement \( u \) and the slope \( dw/dr \) for \( r = c \) can be calculated. Thus all the quantities on the right-hand side of Eqs. (231) are known, and the values of \( d^2u/dr^2 \) and of \( d^3w/dr^3 \) for \( r = c \) can be calculated. As soon as these values are known, another radial step of length \( c \) can be made, and all the quantities entering in the right-hand side of Eqs. (231) can be calculated for \( r = 2c^* \) and so on. The numerical

1 The direction for \( Q_r \) is opposite to that used in Fig. 28. This explains the minus sign in Eq. (e).

* If the intervals into which the radius is divided are sufficiently small, a simple procedure, such as that used in S. Timoshenko's "Vibration Problems in Engineering," 3d ed., p. 143, can be applied. The numerical results represented in Fig. 201 are
values of $u$ and $w$ and their derivatives at the end of any interval being known, the values of the forces $N_r$ and $N_t$ can then be calculated from Eqs. (c) and the bending moments $M_r$ and $M_t$ from Eqs. (52) and (53) (see page 52). By such repeated calculations we proceed up to the radial distance $r = a$ at which the radial force $N_r$ vanishes. In this way we obtain a circular plate of radius $a$ bent by moments $M_0$ uniformly distributed along the edge. By changing the numerical values of $\epsilon_0$ and $1/\rho_0$ at the center we obtain plates with various values of the outer radius and various values of the moment along the edge.

Figure 201 shows graphically the results obtained for a plate with

$$a \approx 23h \quad \text{and} \quad (M_r)_{r=a} = M_0 = 2.93 \cdot 10^{-3} \frac{D}{h}$$

It will be noted that the maximum deflection of the plate is $0.55h$, which is about 9 per cent less than the deflection $w_0$ given by the elementary theory which neglects the strain in the middle plane of the plate. The forces $N_r$ and $N_t$ are both positive in the central portion of the plate. In the outer portion of the plate the forces $N_t$ become negative; i.e., obtained in this manner. A higher accuracy can be obtained by using the methods of Adams or Störmer. For an account of the Adams method see Francis Bashforth’s book on forms of fluid drops, Cambridge University Press, 1883. Störmer’s method is discussed in detail in A. N. Krilov’s book “Approximate Calculations,” published by the Russian Academy of Sciences, Moscow, 1935. See also L. Collatz, “Numerische Behandlung von Differentialgleichungen,” Berlin, 1951.
compression exists in the tangential direction. The maximum tangential compressive stress at the edge amounts to about 18 per cent of the maximum bending stress $6M_0/h^2$. The bending stresses produced by the moments $M_r$ and $M_t$ are somewhat smaller than the stress $6M_0/h^2$ given by the elementary theory and become smallest at the center, at which point the error of the elementary theory amounts to about 12 per cent. From this numerical example it may be concluded that for deflections of the order of $0.5h$ the errors in maximum deflection and maximum stress as given by the elementary theory become considerable and that the strain of the middle plane must be taken into account to obtain more accurate results.

97. Approximate Formulas for Uniformly Loaded Circular Plates with Large Deflections. The method used in the preceding article can also be applied in the case of lateral loading of a plate. It is not, however, of practical use, since a considerable amount of numerical calculation is required to obtain the deflections and stresses in each particular case. A more useful formula for an approximate calculation of the deflections can be obtained by applying the energy method.\(^1\) Let a circular plate of radius $a$ be clamped at the edge and be subject to a uniformly distributed load of intensity $q$. Assuming that the shape of the deflected surface can be represented by the same equation as in the case of small deflections, we take

$$w = w_0 \left(1 - \frac{r^2}{a^2}\right)^2$$

(a)

The corresponding strain energy of bending from Eq. (m) (page 345) is

$$V = \frac{D}{2} \int_0^{2\pi} \int_0^a \left[ \left(\frac{\partial^2 w}{\partial r^2}\right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial r}\right)^2 + 2\nu \frac{\partial w}{\partial r} \frac{\partial^2 w}{\partial r^2}\right] r \, dr \, d\theta = \frac{32\pi}{3} \frac{w_0^2}{a^2} D$$

(b)

For the radial displacements we take the expression

$$u = r(a - r)(C_1 + C_2r + C_3r^2 + \cdots)$$

(c)

each term of which satisfies the boundary conditions that $u$ must vanish at the center and at the edge of the plate. From expressions (a) and (c) for the displacements, we calculate the strain components $\epsilon_r$ and $\epsilon_t$ of the middle plane as shown in the preceding article and obtain the strain energy due to stretching of the middle plane by using the expression

$$V_1 = 2\pi \int_0^a \left(\frac{N_r \epsilon_r}{2} + \frac{N_t \epsilon_t}{2}\right) r \, dr = \frac{\pi Eh}{2} \int_0^a (\epsilon_r^2 + \epsilon_t^2 + 2\nu \epsilon_r \epsilon_t) r \, dr$$

(d)

\(^1\) See Timoshenko, "Vibration Problems," p. 452. For approximate formulas see also Table 82.
Taking only the first two terms in series (c), we obtain
\[ V_1 = \frac{\pi Eha^2}{1 - \nu^2} \left( 0.250C_1^2a^2 + 0.1167C_2^2a^4 + 0.300C_1C_2a^3 - 0.00846C_1a^2 \frac{8w_0^2}{a^2} + 0.00682C_2a^2 \frac{8w_0^2}{a^2} + 0.00477 \frac{64w_0^4}{a^4} \right) \] (e)

The constants \( C_1 \) and \( C_2 \) are now determined from the condition that the total energy of the plate for a position of equilibrium is a minimum. Hence
\[ \frac{\partial V_1}{\partial C_1} = 0 \quad \text{and} \quad \frac{\partial V_1}{\partial C_2} = 0 \] (f)

Substituting expression (e) for \( V_1 \), we obtain two linear equations for \( C_1 \) and \( C_2 \). From these we find that
\[ C_1 = 1.185 \frac{w_0^2}{a^2} \quad \text{and} \quad C_2 = -1.75 \frac{w_0^2}{a^2} \]

Then, from Eq. (e) we obtain\(^1\)
\[ V_1 = 2.59\pi D \frac{w_0^4}{a^2h^2} \] (g)

Adding this energy, which results from stretching of the middle plane, to the energy of bending (b), we obtain the total strain energy
\[ V + V_1 = \frac{32}{3} \pi D \frac{w_0^2}{a^2} \left( 1 + 0.244 \frac{w_0^2}{h^2} \right) \] (h)

The second term in the parentheses represents the correction due to strain in the middle surface of the plate. It is readily seen that this correction is small and can be neglected if the deflection \( w_0 \) at the center of the plate is small in comparison with the thickness \( h \) of the plate.

The strain energy being known from expression (h), the deflection of the plate is obtained by applying the principle of virtual displacements. From this principle it follows that
\[ \frac{d(V + V_1)}{dw_0} \delta w_0 = 2\pi \int_0^a q \delta w r \, dr = 2\pi q \delta w_0 \int_0^a \left( 1 - \frac{r^2}{a^2} \right)^2 r \, dr \]

Substituting expression (h) in this equation, we obtain a cubic equation for \( w_0 \). This equation can be put in the form
\[ w_0 = \frac{qa^4}{64D} \frac{1}{1 + 0.488 \frac{w_0^2}{h^2}} \] (232)

The last factor on the right-hand side represents the effect of the stretching of the middle surface on the deflection. Because of this effect the deflection \( w_0 \) is no longer proportional to the intensity \( q \) of the load, and

\(^1\) It is assumed that \( \nu = 0.3 \) in this calculation.
the rigidity of the plate increases with the deflection. For example, taking \( w_0 = \frac{1}{2} h \), we obtain, from Eq. (232),

\[
w_0 = 0.89 \frac{qa^4}{64D}
\]

This indicates that the deflection in this case is 11 per cent less than that obtained by neglecting the stretching of the middle surface.

Up to now we have assumed the radial displacements to be zero on the periphery of the plate. Another alternative is to assume the edge as free to move in the radial direction. The expression (232) then has to be replaced by

\[
w_0 = \frac{qa^4}{64D} \frac{1}{1 + 0.146 \frac{w_0^2}{h^2}}
\]

(233)

This results\(^1\) which shows that under the latter assumption the effect of the stretching of the plate is considerably less marked than under the former one. Taking, for instance, \( w_0 = \frac{1}{2} h \), we arrive at \( w_0 = 0.965 \frac{(qa^4)}{64D} \), with an effect of stretching of only \( 3.4 \) per cent in place of 11 per cent obtained above.

Furthermore we can conclude from Eqs. (b) and (c) of Art. 96 that, if \( N_r = 0 \) on the edge, then the edge value of \( N_t \) becomes \( N_t = \frac{Eh}{r} \), that is, negative. We can expect, therefore, that for a certain critical value of the lateral load the edge zone of the plate will become unstable.\(^2\)

Another method for the approximate solution of the problem has been developed by A. Nádai.\(^3\) He begins with equations of equilibrium similar to Eqs. (231). To derive them we have only to change Eq. (f), of the preceding article, to fit the case of lateral load of intensity \( q \). After such a change the expression for the shearing force evidently becomes

\[
Q_r = -N_r \frac{dw}{dr} - \frac{1}{r} \int_0^r qr \, dr
\]

(1)

Using this expression in the same manner in which expression (f) was used in the preceding article, we obtain the following system of equations in place of Eqs. (231):

\[
\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = -\frac{1 - v}{2r} \left( \frac{dw}{dr} \right)^2 - \frac{dw}{dr} \frac{d^2 w}{dr^2}
\]

\[
\frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} = \frac{12}{h^2} \frac{dw}{dr} \left[ \frac{du}{dr} + \frac{v}{r} \frac{u}{r} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 \right] + \frac{1}{D} \int_0^r qr \, dr
\]

(234)

\(^1\) Obtained by a method which will be described in Art. 100.

\(^2\) The instability occurring in such a case has been investigated by D. Y. Panov and V. I. Feodosiev, Priklad. Mat. Mekhan., vol. 12, p. 389, 1948.

To obtain an approximate solution of the problem a suitable expression for the deflection $w$ should be taken as a first approximation. Substituting it in the right-hand side of the first of the equations (234), we obtain a linear equation for $u$ which can be integrated to give a first approximation for $u$. Substituting the first approximations for $u$ and $w$ in the right-hand side of the second of the equations (234), we obtain a linear differential equation for $w$ which can be integrated to give a second approximation for $w$. This second approximation can then be used to obtain further approximations for $u$ and $w$ by repeating the same sequence of calculations.

In discussing bending of a uniformly loaded circular plate with a clamped edge, Nádai begins with the derivative $dw/dr$ and takes as first approximation the expression
\[
\frac{dw}{dr} = C \left[ \frac{r}{a} - \left( \frac{r}{a} \right)^n \right] \tag{j}
\]
which vanishes for $r = 0$ and $r = a$ in compliance with the condition at the built-in edge. The first of the equations (234) then gives the first approximation for $u$. Substituting these first approximations for $u$ and $dw/dr$ in the second of the equations (234) and solving it for $q$, we determine the constants $C$ and $n$ in expression (j) so as to make $q$ as nearly a constant as possible. In this manner the following equation\(^1\) for calculating the deflection at the center is obtained when $\nu = 0.25$:
\[
\frac{w_0}{h} + 0.583 \left( \frac{w_0}{h} \right)^3 = 0.176 \frac{q}{E} \left( \frac{a}{h} \right)^4
\tag{235}
\]

In the case of very thin plates the deflection $w_0$ may become very large in comparison with $h$. In such cases the resistance of the plate to bending can be neglected, and it can be treated as a flexible membrane. The general equations for such a membrane are obtained from Eqs. (234) by putting zero in place of the left-hand side of the second of the equations. An approximate solution of the resulting equations is obtained by neglecting the first term on the left-hand side of Eq. (235) as being small in comparison with the second term. Hence
\[
0.583 \left( \frac{w_0}{h} \right)^3 \approx 0.176 \frac{q}{E} \left( \frac{a}{h} \right)^4 \quad \text{and} \quad w_0 = 0.665a \frac{3\sqrt{qa}}{Eh}
\]

\(^1\) Another method for the approximate solution of Eqs. (234) was developed by K. Federhofer, *Eisenbau*, vol. 9, p. 152, 1918; see also *Forschungsarb. VDI*, vol. 7, p. 148, 1936. His equation for $w_0$ differs from Eq. (235) only by the numerical value of the coefficient on the left-hand side; viz., 0.523 must be used instead of 0.583 for $\nu = 0.25$. 
A more complete investigation of the same problem gives

\[ w_0 = 0.662a \frac{3q\alpha}{\sqrt{Eh}} \]  \hspace{1cm} (236)

This formula, which is in very satisfactory agreement with experiments, shows that the deflections are not proportional to the intensity of the load but vary as the cube root of that intensity. For the tensile stresses at the center of the membrane and at the boundary the same solution gives, respectively,

\[ (\sigma_r)_{r=0} = 0.423 \frac{3E^2q^2a^2}{h^2} \quad \text{and} \quad (\sigma_r)_{r=a} = 0.328 \frac{3E^2q^2a^2}{h^2} \]

To obtain deflections that are proportional to the pressure, as is often required in various measuring instruments, recourse should be had to corrugated membranes such as that shown in Fig. 202. As a result of the corrugations the deformation consists primarily in bending and thus increases in proportion to the pressure. If the corrugation (Fig. 202) follows a sinusoidal law and the number of waves along a diameter is sufficiently large \((n > 5)\) then, with the notation of Fig. 186, the following expression for \(w_0 = (w)_{\text{max}}\) may be used:

\[ 8 \left( \frac{w_0}{h} \right) \left[ \frac{2}{3(1 - \nu^2)} + \left( \frac{a}{h} \right)^2 \right] + \frac{6}{7} \left( \frac{w_0}{h} \right)^3 = \frac{q}{E} \left( \frac{a}{h} \right)^4 \]

98. Exact Solution for a Uniformly Loaded Circular Plate with a Clamped Edge. To obtain a more satisfactory solution of the problem of large deflections of a uniformly loaded circular plate with a clamped edge, it is necessary to solve Eqs. (234). To do this we first write the equations in a somewhat different form. As may be seen from its deri-

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3 The theory of deflection of such membranes is discussed by K. Stange, \textit{Ingr.-Arch.}, vol. 2, p. 47, 1931.

4 For a bibliography on diaphragms used in measuring instruments see M. D. Hersey's paper in \textit{NACA Rep.} 165, 1923.


6 This solution is due to S. Way, \textit{Trans. ASME}, vol. 56, p. 627, 1934.
vation in Art. 96, the first of these equations is equivalent to the equation

\[ N_r - N_t + r \frac{dN_r}{dr} = 0 \]  

(237)

Also, as is seen from Eq. (e) of Art. 96 and Eq. (i) of Art. 97, the second of the same equations can be put in the following form:

\[ D \left( \frac{d^3w}{dr^2} + \frac{1}{r} \frac{d^2w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} \right) = N_r \frac{dw}{dr} + \frac{qr}{2} \]  

(238)

From the general expressions for the radial and tangential strain (page 396) we obtain

\[ \epsilon_r = \epsilon_t + r \frac{d\epsilon_t}{dr} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 \]

Substituting

\[ \epsilon_r = \frac{1}{kE}(N_r - \nu N_t) \quad \text{and} \quad \epsilon_t = \frac{1}{kE}(N_t - \nu N_r) \]

in this equation and using Eq. (237), we obtain

\[ r \frac{d}{dr} (N_r + N_t) + \frac{hE}{2} \left( \frac{dw}{dr} \right)^2 = 0 \]  

(239)

The three Eqs. (237), (238), and (239) containing the three unknown functions \( N_r, N_t, \) and \( w \) will now be used in solving the problem. We begin by transforming these equations to a dimensionless form by introducing the following notations:

\[ p = \frac{q}{E} \quad \xi = \frac{r}{h} \quad S_r = \frac{N_r}{hE} \quad S_t = \frac{N_t}{hE} \]  

(240)

With this notation, Eqs. (237), (238), and (239) become, respectively,

\[ \frac{d}{d\xi} (\xi S_r) - S_t = 0 \]  

(241)

\[ \frac{1}{12(1 - \nu^2)} \frac{d}{d\xi} \left[ \frac{1}{\xi} d\xi \left( \xi \frac{dw}{dr} \right) \right] = \frac{p\xi}{2} + S_r \frac{dw}{dr} \]  

(242)

\[ \xi \frac{d}{d\xi} (S_r + S_t) + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 = 0 \]  

(243)

The boundary conditions in this case require that the radial displacement \( u \) and the slope \( dw/dr \) vanish at the boundary. Using Eq. (b) of Art. 96 for the displacements \( u \) and applying Hooke's law, these conditions become

\[ (u)_{r=a} = r(S_t - \nu S_r)_{r=a} = 0 \]  

(244)

\[ \left( \frac{dw}{dr} \right)_{r=a} = 0 \]  

(a)
Assuming that \( S_r \) is a symmetrical function and \( dw/dr \) an antisymmetrical function of \( \xi \), we represent these functions by the following power series:

\[
S_r = B_0 + B_2 \xi^2 + B_4 \xi^4 + \cdots \quad (b)
\]

\[
\frac{dw}{dr} = \sqrt{8} \left( C_1 \xi + C_3 \xi^3 + C_5 \xi^5 + \cdots \right) \quad (c)
\]

in which \( B_0, B_2, \ldots \) and \( C_1, C_3, \ldots \) are constants to be determined later. Substituting the first of these series in Eq. (241), we find

\[
S_r = B_0 + 3B_2 \xi^2 + 5B_4 \xi^4 + \cdots \quad (d)
\]

By integrating and differentiating Eq. (c), we obtain, respectively,

\[
\frac{w}{h} = \sqrt{8} \left( \frac{C_1 \xi^2}{2} + \frac{C_3 \xi^4}{4} + \frac{C_5 \xi^6}{6} + \cdots \right) \quad (e)
\]

\[
\frac{d}{d\xi} \left( \frac{dw}{dr} \right) = \sqrt{8} \left( C_1 + 3C_3 \xi^2 + 5C_5 \xi^4 + \cdots \right) \quad (f)
\]

It is seen that all the quantities in which we are interested can be found if we know the constants \( B_0, B_2, \ldots, C_1, C_3, \ldots \). Substituting series (b), (c), and (d) in Eqs. (242) and (243) and observing that these equations must be satisfied for any value of \( \xi \), we find the following relations between the constants \( B \) and \( C \):

\[
B_k = -\frac{4}{k(k+2)} \sum_{m=1,3,5,\ldots}^{k-1} C_mC_{k-m} \quad k = 2, 4, 6, \ldots
\]

\[
C_k = \frac{12(1-v^2)}{k^2 - 1} \sum_{m=0,2,4,\ldots}^{k-3} B_mC_{k-2-m} \quad k = 5, 7, 9, \ldots \quad (g)
\]

\[
C_3 = \frac{3}{2} \left( 1 - v^2 \right) \left( \frac{p}{2 \sqrt{8}} + B_0 C_1 \right)
\]

It can be seen that when the two constants \( B_0 \) and \( C_1 \) are assigned, all the other constants are determined by relations (g). The quantities \( S_r, S_t \), and \( dw/dr \) are then determined by series (b), (d), and (c) for all points in the plate. As may be seen from series (b) and (f), fixing \( B_0 \) and \( C_1 \) is equivalent to selecting the values of \( S_r \) and the curvature at the center of the plate.\(^1\)

To obtain the following curves for calculating deflections and stresses in particular cases, the procedure used was: For given values of \( v \) and

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\(^1\)The selection of these same quantities has already been encountered in the case of bending of circular plates by moments uniformly distributed along the edge (see page 398).
\( p = q/E \) and for selected values of \( B_0 \) and \( C_1 \), a considerable number of numerical cases were calculated,\(^1\) and the radii of the plates were determined so as to satisfy the boundary condition (a). For all these plates the values of \( S_r \) and \( S_t \) at the boundary were calculated, and the values of the radial displacements \((u)_r^{\infty}\) at the boundary were determined. Since all calculations were made with arbitrarily assumed values of \( B_0 \) and \( C_1 \), the boundary condition (244) was not satisfied. However, by interpolation it was possible to obtain all the necessary data for plates for which both conditions (244) and (a) are satisfied. The results of these calculations are represented graphically in Fig. 203. If the deflection of the plate is found from this figure, the corresponding stress can be obtained by using the curves of Fig. 204. In this figure, curves are given for the membrane stresses

\[ \sigma_r = \frac{N_r}{h} \]

and for the bending stresses

\[ \sigma_r' = \frac{6M_r}{h^3} \]

as calculated for the center and for the edge of the plate.\(^2\) By adding together \( \sigma_r \) and \( \sigma_r' \), the total maximum stress at the center and at the edge of the plate can be obtained. For purposes of comparison Figs. 203 and 204 also include straight lines showing the results obtained from

\(^1\) Nineteen particular cases have been calculated by Way, *op. cit.*

\(^2\) The stresses are given in dimensionless form.
the elementary theory in which the strain of the middle plane is neglected. It will be noted that the errors of the elementary theory increase as the load and deflections increase.

![Diagram of stress and strain in a plate](image)

**Fig. 204**

99. A Simply Supported Circular Plate under Uniform Load. An exact solution of the problem\(^1\) can be obtained by a series method similar to that used in the preceding article.

Because of the axial symmetry we have again \(dw/dr = 0\) and \(N_r = N_t\) at \(r = 0\). Since the radial couples must vanish on the edge, a further condition is

\[
\left[ \frac{d}{dr} \left( \frac{dw}{dr} \right) + \frac{v}{r} \frac{dw}{dr} \right]_{r=0} = 0
\]

With regard to the stress and strain in the middle plane of the plate two boundary conditions may be considered:

1. Assuming the edge is immovable we have, by Eq. (244), \(S_t - \nu S_r = 0\), which, by Eq. (237), is equivalent to

\[
\left[ S_r (1 - \nu) + r \frac{dS_r}{dr} \right]_{r=a} = 0
\]

2. Supposing the edge as free to move in the radial direction we simply have

\[(S_r)_{r=a} = 0\]  

(c)

The functions \(S_r\) and \(dw/dr\) may be represented again in form of the series

\[S_r = \frac{h^2}{12(1 - \nu^2)\alpha^2 r} (B_{1\rho} + B_{2\rho^2} + B_{3\rho^4} + \cdots)\]  

(d)

\[
\frac{dw}{dr} = -\frac{h}{2a \sqrt{3}} (C_{1\rho} + C_{2\rho^2} + C_{4\rho^4} + \cdots)
\]  

(e)

where \(\rho = r/a\). Using these series and also Eqs. (241), (242), (243), from which the quantity \(S_r\) can readily be eliminated, we arrive at the following relations between the constants \(B\) and \(C\):

\[B_k = -\frac{1 - \nu^2}{2(k^2 - 1)} \sum_{m=1,3,5,\ldots}^{k-2} C_mC_{k-m-1} \quad k = 3, 5, \ldots\]  

(f)

\[C_k = \frac{1}{k^2 - 1} \sum_{m=1,3,5,\ldots}^{k-2} C_mB_{k-m-1} \quad k = 5, 7, \ldots\]  

(g)

\[8C_1 - B_1C_1 + 12 \sqrt{3} (1 - \nu^2) \frac{pa^4}{h^4} = 0\]  

(h)

where \(p = q/E\), \(q\) being the intensity of the load.

Again, all constants can easily be expressed in terms of both constants \(B_1\) and \(C_1\), for which two additional relations, ensuing from the boundary conditions, hold:

In case 1 we have

\[\sum_{k=1,3,5,\ldots} B_k(k - \nu) = 0 \quad \sum_{k=1,3,5,\ldots} C_k(k + \nu) = 0\]  

(i)

and in case 2

\[\sum_{k=1,3,5,\ldots} B_k = 0 \quad \sum_{k=1,3,5,\ldots} C_k(k + \nu) = 0\]  

(j)

To start the resolution of the foregoing system of equations, suitable values of \(B_1\) and \(C_1\) may be taken on the basis of an approximate solution. Such a solution, satisfying condition (a), can be, for instance, of the form

\[\frac{dw}{dr} = C(\beta \rho^n - \rho)\]  

(k)

where \(C\) is a constant and \(\beta = \frac{1 + \nu}{n + \nu} (n = 3, 5, \ldots)\). Substituting this in Eqs. (241) and (243), in which \(\xi\) must be replaced by \(\rho a/h\), and eliminating \(S_r\) we obtain

\[S_r = c_1 + \frac{c_2}{\rho^2} - \frac{C^2}{2} \left(\beta^2 \frac{\rho^{2n}}{n_1} - 2\beta \frac{\rho^{n+1}}{n_2} + \frac{\rho^4}{8}\right)\]  

(l)
Table 82. Data for Calculation of Approximate Values of Deflections $w_0$ and Stresses in Uniformly Loaded Plates

$\nu = 0.3$

<table>
<thead>
<tr>
<th>Boundary Conditions</th>
<th>A</th>
<th>B</th>
<th>Center</th>
<th>Edge</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha_r = \alpha_t$</td>
<td>$\beta_r = \beta_t$</td>
<td>$\alpha_r$</td>
<td>$\alpha_t$</td>
</tr>
<tr>
<td>Plate clamped</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Edge immovable</td>
<td>0.471</td>
<td>0.171</td>
<td>0.976</td>
<td>2.86</td>
</tr>
<tr>
<td>Edge free to move</td>
<td>0.146</td>
<td>0.171</td>
<td>0.500</td>
<td>2.86</td>
</tr>
<tr>
<td>Plate simply supported</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Edge immovable</td>
<td>1.852</td>
<td>0.696</td>
<td>0.905</td>
<td>1.778</td>
</tr>
<tr>
<td>Edge free to move</td>
<td>0.262</td>
<td>0.696</td>
<td>0.295</td>
<td>1.778</td>
</tr>
</tbody>
</table>
Herein \( c_1 \) and \( c_2 \) are constants of integration and

\[
\begin{align*}
  n_1 &= 4n(n + 1) \\
  n_2 &= (n + 1)(n + 3)
\end{align*}
\]

Let us, for example, assume the boundary conditions of case 2. Then we obtain

\[
c_1 = \frac{C^2}{2} \left( \frac{\beta^2}{n_1} - \frac{2\beta}{n_2} + \frac{1}{8} \right) \quad c_2 = 0 \quad (m)
\]

The constant \( C \), finally, can be determined by some strain energy method—for example, that described in Art. 100. Using there Eqs. (m) or (o) we have only to replace \( d\varphi/dr \) and \( dw/dr \) by approximate expressions in accordance with Eqs. (k) and (l) given above.

The largest values of deflections and of total stresses obtained by Federhofer and Egger from the exact solution are given in Fig. 205 for case 1 and in Fig. 206 for case 2. The calculation has been carried out for \( \nu = 0.25 \).

Table 82 may be useful for approximate calculations of the deflection \( \omega \) at the
center, given by an equation of the form
\[
\frac{w_0}{h} + A \left( \frac{w_0}{h} \right)^3 = B \left( \frac{u}{E} \right) \left( \frac{a}{h} \right)^4 \quad (a)
\]
also of the stresses in the middle plane, given by
\[
\sigma_r = \alpha_r E \frac{w_0^2}{a^2} \quad \sigma_t = \alpha_t E \frac{w_0^2}{a^2} \quad (o)
\]
and of the extreme fiber bending stresses
\[
\sigma'_r = \beta_r E \frac{w_0 h}{a^2} \quad \sigma'_t = \beta_t E \frac{w_0 h}{a^2} \quad (p)
\]

100. Circular Plates Loaded at the Center. An approximate solution of this problem can be obtained by means of the method described in Art. 81.

The work of the internal forces corresponding to some variation \( \delta \epsilon_r, \delta \epsilon_t \) of the strain is
\[
\delta V_1 = -2\pi \int_0^a (N_r \delta \epsilon_r + N_t \delta \epsilon_t) r \, dr
\]

Using Eqs. (a) and (b) of Art. 96 we have
\[
\delta V_1 = -2\pi \int_0^a \left\{ N_r \delta \left[ \frac{du}{dr} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 \right] + N_t \delta \left( \frac{u}{r} \right) \right\} r \, dr \quad (a)
\]

We assume, furthermore, that either the radial displacements in the middle plane or the radial forces \( N_r \) vanish on the boundary. Then, integrating expression (a) by parts and putting \( \delta u = 0 \) or \( N_r = 0 \) on \( r = a \), we obtain
\[
\delta V_1 = 2\pi \int_0^a \left[ \frac{d}{dr} \left( r N_r \right) - N_t \right] \delta u \, dr - 2\pi \int_0^a r N_r \frac{dw}{dr} \delta \left( \frac{dw}{dr} \right) \, dr \quad (b)
\]
The work of the bending moments \( M_r \) and \( M_t \) on the variation \( \delta (-d^2 w/dr^2) \) and \( \delta (-\frac{1}{2} \frac{dw}{dr}) \) of the curvatures is similarly
\[
\delta V_2 = 2\pi \int_0^a \left[ M_r \delta \left( \frac{d^2 w}{dr^2} \right) + M_t \delta \left( \frac{1}{r} \frac{dw}{dr} \right) \right] r \, dr \quad (c)
\]
Now we suppose that either the radial bending moment \( M_r \) or the slope \( \delta (dw/dr) \) becomes zero on the boundary. Integration of expression (c) by parts then yields
\[
\delta V_2 = 2\pi \int_0^a D \frac{d}{dr} (\Delta w) \delta \left( \frac{dw}{dr} \right) r \, dr \quad (d)
\]
Finally, the work of the external forces is
\[
\delta V_3 = 2\pi \int_0^a q \delta w \, r \, dr
\]
or, by putting
\[
\psi = \frac{1}{r} \int_0^r q r \, dr \quad q = \frac{1}{r} \frac{d}{dr} (r \psi) \quad (e)
\]

\(^1\) The sign is negative if the bottom of the plate is in compression.
we have
\[ \delta V_1 = 2\pi \int_0^a \frac{d}{dr} (r\psi) \delta w \, r \, dr \]
Provided \( \delta w = 0 \) on the boundary we finally obtain
\[ \delta V_2 = -2\pi \int_0^a r\psi \delta \left( \frac{dw}{dr} \right) r \, dr \]  \( \text{(f)} \)

The condition \( \delta(V_1 + V_2 + V_3) = 0 \) now yields the equation
\[ \int_0^a \left[ D \frac{d}{dr} (\Delta w) - \psi - N_r \frac{dw}{dr} \right] \frac{d}{dr} (\delta w) r \, dr + \int_0^a \left[ \frac{d}{dr} (rN_r) - N_t \right] \delta u \, dr = 0 \]  \( \text{(g)} \)

We could proceed next by assuming both variations \( \delta w \) and \( \delta u \) as arbitrary. Thus we would arrive at the second of the differential equations (234), \( N_r \) being given by expression (c) of Art. 96, and at Eq. (d) of the same article. If we suppose only this latter equation of equilibrium to be satisfied, then we have still to fulfill the condition
\[ \int_0^a \left[ D \frac{d}{dr} (\Delta w) - \psi - \frac{1}{r} \frac{df}{dr} \frac{dw}{dr} \right] \frac{d}{dr} (\delta w) r \, dr = 0 \]  \( \text{(h)} \)
in which \( f \) is a stress function defining
\[ N_r = \frac{1}{r} \frac{df}{dr} \quad N_t = \frac{d^2 f}{dr^2} \]  \( \text{(i)} \)
and governed by the differential equation
\[ \frac{d}{dr} (\Delta f) = -\frac{Eh}{2r} \left( \frac{dw}{dr} \right)^2 \]  \( \text{(j)} \)
which follows from Eq. (239). Integrating expression (h) by parts once more we obtain
\[ \int_0^a \left[ D\Delta w - q - \frac{1}{r} \frac{d}{dr} \left( \frac{df}{dr} \frac{dw}{dr} \right) \right] \delta w \, r \, dr = 0 \]  \( \text{(k)} \)

With intent to use the method described in Art. 81 we take the deflection in the form
\[ w = a_1 \varphi_1(r) + a_2 \varphi_2(r) + \cdots + a_n \varphi_n(r) \]  \( \text{(l)} \)
Just as in the case of the expression (211) each function \( \varphi_i(r) \) has to satisfy two boundary conditions prescribed for the deflection. Substituting expression (l) either in Eq. (h) or in Eq. (k) and applying the same reasoning as in Art. 81, we arrive at a sequence of equations of the form
\[ \int_0^a X \frac{d\varphi_i}{dr} r \, dr \quad i = 1, 2, \ldots, n \]  \( \text{(m)} \)
in which
\[ X = D \frac{d}{dr} (\Delta w) - \psi - \frac{1}{r} \frac{df}{dr} \frac{dw}{dr} \]  \( \text{(n)} \)
or at a set of equations

\[ \int_0^a Y \varphi_i r \, dr \quad i = 1, 2, \ldots, n \]  

(a)

where

\[ Y = D\Delta \Delta w - q - \frac{1}{r} \frac{d}{dr} \left( \frac{df}{dr} \frac{dw}{dr} \right) \]  

(p)

Now let us consider a clamped circular plate with a load \( P \) concentrated at \( r = 0 \). We reduce expression (l) to its first term by taking the deflection in the form

\[ w = w_0 \left( 1 - \frac{r^2}{a^2} + 2 \frac{r^2}{a^2} \log \frac{r}{a} \right) \]  

(q)

which holds rigorously for a plate with small deflections. From Eq. (j) we obtain, by integration,

\[ \frac{df}{dr} = - \frac{Ehw_0^2 r^2}{a^4} \left( \log^2 \frac{r}{a} - \frac{3}{2} \frac{r}{a} \log \frac{r}{a} + \frac{7}{8} \right) + C_1 r + C_2 \frac{r}{r} \]  

(r)

Let there be a free radial displacement at the boundary. The constants of integration \( C_1 \) and \( C_2 \) then are determined by two conditions. The first, namely,

\[ (N_r)_{r=a} = 0 \]

can be rewritten as

\[ \left( \frac{1}{r} \frac{df}{dr} \right)_{r=a} = 0 \]  

(s)

and the second is

\[ \left( \frac{df}{dr} \right)_{r=0} = 0 \]  

(t)

This latter condition must be added in order to limit, at \( r = 0 \), the value of the stress \( N_r \) given by Eq. (i). Thus we obtain

\[ C_1 = \frac{7}{8} \frac{Ehw_0^2}{a^2} \quad C_2 = 0 \]

The load function is equal to

\[ \psi = \frac{P}{2\pi r} \]

in our case, and expressions (q) and (r) yield

\[ X = D \frac{8w_0}{a^2} - \frac{P}{2\pi r} + \frac{4Ew_0^2 h}{a^3} \left( \frac{r^3}{a^3} \log \frac{r}{a} - \frac{3}{2} \frac{r}{a} \log \frac{r}{a} + \frac{7}{8} \frac{r^3}{a^3} \log \frac{r}{a} - \frac{7}{8} \frac{r}{a} \log \frac{r}{a} \right) \]  

(u)

while \( \varphi \) is given by the expression in the parentheses in Eq. (q). Substituting this in Eq. (m) we arrive at the relation

\[ 16 Dw_0 + \frac{191}{648} Ehw_0^3 = \frac{Pa^2}{\pi} \]  

(v)
The general expressions for the extreme fiber bending stresses corresponding to the
deflection \((q)\) and obtainable by means of Eqs. (101) are

\[
\sigma_r' = \frac{2Ehv_0}{(1 - \nu^2)a^2} \left[ (1 + \nu) \log \frac{a}{r} - 1 \right]
\]

\[
\sigma_r'' = \frac{2Ehv_0}{(1 - \nu^2)a^2} \left[ (1 + \nu) \log \frac{a}{r} - \nu \right]
\]

These expressions yield infinite values of stresses as \(r\) tends to zero. However, assum-
ing the load \(P\) to be distributed uniformly over a circular area with a small radius
\(r = c\), we can use a simple relation existing in plates with small deflections between
the stresses \(\sigma_r'' = \sigma_i''\) at the center of such an area and the stresses \(\sigma_r' = \sigma_i'\) caused at
\(r = c\) by the same load \(P\) acting at the point \(r = 0\). According to Nádai's result,
expressed in terms of stresses,

\[
\sigma_r'' = \sigma_i'' = \sigma_r' + \frac{3P}{2\pi h^2}
\]

Applying this relation to the plate with large deflections we obtain, at the center of
the loaded area with a radius \(c\), approximately

\[
\sigma_r'' = \sigma_i'' = \frac{2Ehv_0}{(1 - \nu^2)a^2} \left[ (1 + \nu) \log \frac{a}{c} - 1 \right] + \frac{3P}{2\pi h^2}
\]

The foregoing results hold for a circular plate with a clamped and movable edge. By
introducing other boundary conditions we obtain for \(w_0\) an equation

\[
\frac{w_0}{h} + A \left( \frac{w_0}{h} \right)^3 = B \frac{Pc^2}{Eh^4}
\]

which is a generalization of Eq. \((v)\). The constants \(A\) and \(B\) are given in Table 83. The
same table contains several coefficients \(^2\) needed for calculation of stresses

\[
\sigma_r = \alpha_r E \frac{w_0^2}{a^2} \quad \sigma_i = \alpha_i E \frac{w_0^2}{a^2}
\]

acting in the middle plane of the plate and the extreme fiber bending stresses

\[
\sigma_r' = \beta_r E \frac{wh}{a^2} \quad \sigma_i' = \beta_i E \frac{wh}{a^2}
\]

The former are calculated using expressions \((i)\), the latter by means of expressions
(101) for the moments, the sign being negative if the compression is at the bottom.\(^3\)

101. General Equations for Large Deflections of Plates. In discussing
the general case of large deflections of plates we use Eq. (219), which was

\(^1\) A. Nádai, "Elastische Platten," p. 63, Berlin, 1925.
\(^2\) All data contained in Table 82 are taken from A. S. Volmir, \textit{op. cit.}
\(^3\) For bending of the ring-shaped plates with large deflections see K. Federhofer,
1952, and vol. 11, p. 473, 1953. Large deflections of elliptical plates have been dis-
Table 83. Data for Calculation of Approximate Values of Deflections \( w_0 \) and Stresses in Centrally Loaded Plates 
\( \nu = 0.3 \)

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>A</th>
<th>B</th>
<th>Center ( \alpha_r = \alpha_t )</th>
<th>Edge ( \alpha_r )</th>
<th>( \alpha_t )</th>
<th>( \beta_r )</th>
<th>( \beta_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plate clamped</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Edge immovable</td>
<td>0.443</td>
<td>0.217</td>
<td>1.232</td>
<td>0.357</td>
<td>0.107</td>
<td>-2.198</td>
<td>-0.659</td>
</tr>
<tr>
<td>Edge free to move</td>
<td>0.200</td>
<td>0.217</td>
<td>0.875</td>
<td>0</td>
<td>-0.250</td>
<td>-2.198</td>
<td>-0.659</td>
</tr>
<tr>
<td>Plate simply supported</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Edge immovable</td>
<td>1.430</td>
<td>0.552</td>
<td>0.895</td>
<td>0.488</td>
<td>0.147</td>
<td>0</td>
<td>0.606</td>
</tr>
<tr>
<td>Edge free to move</td>
<td>0.272</td>
<td>0.552</td>
<td>0.407</td>
<td>0</td>
<td>-0.341</td>
<td>0</td>
<td>0.606</td>
</tr>
</tbody>
</table>

Derived by considering the equilibrium of an element of the plate in the direction perpendicular to the plate. The forces \( N_x, N_y \), and \( N_{xy} \) now depend not only on the external forces applied in the \( xy \) plane but also on the strain of the middle plane of the plate due to bending. Assuming that there are no body forces in the \( xy \) plane and that the load is perpendicular to the plate, the equations of equilibrium of an element in the \( xy \) plane are

\[
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \tag{a}
\]

The third equation necessary to determine the three quantities \( N_x, N_y \), and \( N_{xy} \) is obtained from a consideration of the strain in the middle surface of the plate during bending. The corresponding strain components [see Eqs. (221), (222), and (223)] are

\[
\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\
\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \tag{b} \\
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}
\]

By taking the second derivatives of these expressions and combining the
resulting expressions, it can be shown that

\[
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}
\]  

(c)

By replacing the strain components by the equivalent expressions

\[
\varepsilon_x = \frac{1}{hE} (N_x - \nu N_y) \\
\varepsilon_y = \frac{1}{hE} (N_y + \nu N_x) \\
\gamma_{xy} = \frac{1}{hG} N_{xy}
\]

(d)

the third equation in terms of \(N_x, N_y,\) and \(N_{xy}\) is obtained.

The solution of these three equations is greatly simplified by the introduction of a stress function.\(^1\) It may be seen that Eqs. (a) are identically satisfied by taking

\[
N_x = \frac{h}{E} \frac{\partial^2 F}{\partial y^2} \quad N_y = \frac{h}{E} \frac{\partial^2 F}{\partial x^2} \quad N_{xy} = - \frac{h}{E} \frac{\partial^2 F}{\partial x \partial y}
\]

(e)

where \(F\) is a function of \(x\) and \(y\). If these expressions for the forces are substituted in Eqs. (d), the strain components become

\[
\varepsilon_x = \frac{1}{E} \left( \frac{\partial^2 F}{\partial y^2} - \nu \frac{\partial^2 F}{\partial x^2} \right) \\
\varepsilon_y = \frac{1}{E} \left( \frac{\partial^2 F}{\partial x^2} - \nu \frac{\partial^2 F}{\partial y^2} \right) \\
\gamma_{xy} = - \frac{2(1 + \nu)}{E} \frac{\partial^2 F}{\partial x \partial y}
\]

(f)

Substituting these expressions in Eq. (c), we obtain

\[
\frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = E \left[ \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right]
\]

(245)

The second equation necessary to determine \(F\) and \(w\) is obtained by substituting expressions (e) in Eq. (217), which gives

\[
\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{h}{D} \left( q + \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} \right. \\
\left. + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right)
\]

(246)

Equations (245) and (246), together with the boundary conditions, determine the two functions $F$ and $w$. Having the stress function $F$, we can determine the stresses in the middle surface of a plate by applying Eqs. (e). From the function $w$, which defines the deflection surface of the plate, the bending and the shearing stresses can be obtained by using the same formulas as in the case of plates with small deflection [see Eqs. (101) and (102)]. Thus the investigation of large deflections of plates reduces to the solution of the two nonlinear differential equations (245) and (246). The solution of these equations in the general case is unknown. Some approximate solutions of the problem are known, however, and will be discussed in the next article.

In the particular case of bending of a plate to a cylindrical surface whose axis is parallel to the $y$ axis, Eqs. (245) and (246) are simplified by observing that in this case $w$ is a function of $x$ only and that $\partial^2 F/\partial x^2$ and $\partial^2 F/\partial y^2$ are constants. Equation (245) is then satisfied identically, and Eq. (246) reduces to

$$\frac{\partial^4 w}{\partial x^4} = \frac{q}{D} + \frac{N_x}{D} \frac{\partial^2 w}{\partial x^2}$$

Problems of this kind have already been discussed fully in Chap. 1.

If polar coordinates, more convenient in the case of circular plates, are used, the system of equations (245) and (246) assumes the form

$$\Delta \Delta F = -\frac{E}{2} L(w,w)$$

$$\Delta \Delta w = \frac{h}{D} L(w,F) + \frac{q}{D}$$

in which

$$L(w,F) = \frac{\partial^2 w}{\partial r^2} \left( \frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} \right) + \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \frac{\partial^2 F}{\partial r^2} - 2 \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial F}{\partial \theta} \right) \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} \right)$$

and $L(w,w)$ is obtained from the foregoing expression if $w$ is substituted for $F$.

In the case of very thin plates, which may have deflections many times larger than their thickness, the resistance of the plate to bending can be

* These two equations were derived by Th. von Kármán; see "Encyklopädie der Mathematischen Wissenschaften," vol. IV, p. 349, 1910. A general method of nonlinear elasticity has been applied to bending of plates by E. Koppe, Z. angew. Math. Mech., vol. 36, p. 455, 1956.

neglected; i.e., the flexural rigidity $D$ can be taken equal to zero, and the problem reduced to that of finding the deflection of a flexible membrane. Equations (245) and (246) then become

$$\frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = E \left[ \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right]$$

$$\frac{q}{h} + \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} = 0$$

(247)

A numerical solution of this system of equations by the use of finite differences has been discussed by H. Hencky.\(^1\)

The energy method affords another means of obtaining an approximate solution for the deflection of a membrane. The strain energy of a membrane, which is due solely to stretching of its middle surface, is given by the expression

$$V = \frac{1}{2} \iint (N_x \varepsilon_x + N_y \varepsilon_y + N_{xy} \gamma_{xy}) \, dx \, dy$$

$$= \frac{E h}{2(1 - \nu^2)} \iint \left[ \varepsilon_x^2 + \varepsilon_y^2 + 2\nu \varepsilon_x \varepsilon_y + \frac{1}{2}(1 - \nu) \gamma_{xy}^2 \right] \, dx \, dy$$

(248)

Substituting expressions (221), (222), and (223) for the strain components $\varepsilon_x, \varepsilon_y, \gamma_{xy}$, we obtain

$$V = \frac{E h}{2(1 - \nu^2)} \iint \left\{ (\frac{\partial u}{\partial x})^2 + \frac{\partial u}{\partial x} (\frac{\partial w}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + \frac{\partial v}{\partial y} (\frac{\partial w}{\partial y})^2 \right\}$$

$$+ \frac{1}{4} \left[ (\frac{\partial w}{\partial x})^2 + (\frac{\partial w}{\partial y})^2 \right]^2 + 2\nu \left[ \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{1}{2} \frac{\partial v}{\partial y} (\frac{\partial w}{\partial x})^2 + \frac{1}{2} \frac{\partial u}{\partial x} (\frac{\partial w}{\partial y})^2 \right]$$

$$+ \frac{1 - \nu}{2} \left[ (\frac{\partial u}{\partial y})^2 + 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + (\frac{\partial v}{\partial y})^2 + 2 \frac{\partial u}{\partial x} \frac{\partial w}{\partial y} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \right\} \, dx \, dy$$

(249)

In applying the energy method we must assume in each particular case suitable expressions for the displacements $u, v, w$. These expressions must, of course, satisfy the boundary conditions and will contain several arbitrary parameters the magnitudes of which have to be determined by the use of the principle of virtual displacements. To illustrate the method, let us consider a uniformly loaded square membrane\(^3\) with sides of length $2a$ (Fig. 207). The displacements $u, v, w$ in this case must vanish at the boundary. Moreover, from symmetry, it can be concluded

\(^1\) These equations were obtained by A. Föppl, "Vorlesungen über Technische Mechanik," vol. 5, p. 132, 1907.


\(^3\) Calculations for this case are given in the book "Drang und Zwang" by August and Ludwig Föppl, vol. 1, p. 226, 1924; see also Hencky, ibid.
that \( w \) is an even function of \( x \) and \( y \), whereas \( u \) and \( v \) are odd functions of \( x \) and of \( y \), respectively. All these requirements are satisfied by taking the following expressions for the displacements:

\[
\begin{align*}
  w &= w_0 \cos \frac{\pi x}{2a} \cos \frac{\pi y}{2a} \\
  u &= c \sin \frac{\pi x}{a} \cos \frac{\pi y}{2a} \\
  v &= c \sin \frac{\pi y}{a} \cos \frac{\pi x}{2a}
\end{align*}
\]

which contain two parameters \( w_0 \) and \( c \). Substituting these expressions in Eq. (249), we obtain, for \( v = 0.25 \),

\[
V = \frac{Eh}{7.5} \left[ \frac{5\pi^4}{64} \frac{w_0^4}{a^4} - \frac{17\pi^2}{6} \frac{w_0^2}{a^2} + c^2 \left( \frac{35\pi^2}{4} + \frac{80}{9} \right) \right]
\]

The principle of virtual displacements gives the two following equations:

\[
\begin{align*}
  \frac{\partial V}{\partial c} &= 0 \\
  \frac{\partial V}{\partial w_0} \delta w_0 &= \int_{-a}^{a} \int_{-a}^{a} \delta w_0 \cos \frac{\pi x}{2a} \cos \frac{\pi y}{2a} \, dx \, dy
\end{align*}
\]

Substituting expression (h) for \( V \), we obtain from Eq. (i)

\[
c = 0.147 \frac{w_0^3}{a}
\]

and from Eq. (j)

\[
w_0 = 0.802a \sqrt[3]{\frac{3a}{Eh}}
\]

This deflection at the center is somewhat larger than the value (236) previously obtained for a uniformly loaded circular membrane. The tensile strain at the center of the membrane as obtained from expressions (g) is

\[
\epsilon_x = \epsilon_y = \frac{\pi c}{a} = 0.462 \frac{w_0^3}{a^2}
\]

and the corresponding tensile stress is

\[
\sigma = \frac{E}{1-\nu} 0.462 \frac{w_0^3}{a^2} = 0.616 \frac{Ew_0^3}{a^2} = 0.396 \sqrt{\frac{q^2 E a^2}{h^2}}
\]

Some application of these results to the investigation of large deflections of thin plates will be shown in the next article.

1 The right-hand side of Eq. (i) is zero, since the variation of the parameter \( c \) produces only horizontal displacements and the vertical load does not produce work.
102. Large Deflections of Uniformly Loaded Rectangular Plates. We begin with the case of a plate with clamped edges. To obtain an approximate solution of the problem the energy method will be used. The total strain energy $V$ of the plate is obtained by adding to the energy of bending [expression (117), page 88] the energy due to strain of the middle surface [expression (249), page 419]. The principle of virtual displacements then gives the equation

$$
\delta V - \delta \int qw \, dx \, dy = 0 \quad (a)
$$

which holds for any variation of the displacements $u$, $v$, and $w$. By deriving the variation of $V$ we can obtain from Eq. (a) the system of Eqs. (245) and (246), the exact solution of which is unknown. To find an approximate solution of our problem we assume for $u$, $v$, and $w$ three functions satisfying the boundary conditions imposed by the clamped edges and containing several parameters which will be determined by using Eq. (a). For a rectangular plate with sides $2a$ and $2b$ and coordinate axes, as shown in Fig. 207, we shall take the displacements in the following form:

$$
u = (a^2 - x^2)(b^2 - y^2)\left(b_{00} + b_{02}y^2 + b_{20}x^2 + b_{22}x^2y^2\right)
$$

$$
v = (a^2 - x^2)(b^2 - y^2)\left(c_{00} + c_{02}y^2 + c_{20}x^2 + c_{22}x^2y^2\right)
$$

$$
w = (a^2 - x^2)^2(b^2 - y^2)^2(a_{00} + a_{02}y^2 + a_{20}x^2)
$$

The first two of these expressions, which represent the displacements $u$ and $v$ in the middle plane of the plate, are odd functions in $x$ and $y$, respectively, and vanish at the boundary. The expression for $w$, which is an even function in $x$ and $y$, vanishes at the boundary, as do also its first derivatives. Thus all the boundary conditions imposed by the clamped edges are satisfied.

Expressions (b) contain 11 parameters $b_{00}, \ldots, a_{20}$, which will now be determined from Eq. (a), which must be satisfied for any variation of each of these parameters. In such a way we obtain 11 equations, 3 of the form

$$
\frac{\partial}{\partial a_{mn}} \left( V - \int \int qw \, dx \, dy \right) = 0 \quad (c)
$$

and 8 equations of the form\(^2

$$
\frac{\partial V}{\partial b_{mn}} = 0 \quad \text{or} \quad \frac{\partial V}{\partial c_{mn}} = 0 \quad (d)
$$

These equations are not linear in the parameters $a_{mn}$, $b_{mn}$, and $c_{mn}$ as was true in the case of small deflections (see page 344). The three equations of the form (c) will contain terms of the third degree in the parameters $a_{mn}$. Equations of the form (d) will be linear in the parameters $b_{mn}$ and $c_{mn}$ and quadratic in the parameters $a_{mn}$. A solution is obtained by solving Eqs. (d) for the $b_{mn}$'s and $c_{mn}$'s in terms of the $a_{mn}$'s and then substituting these expressions in Eqs. (c). In this way we obtain three equa-


\(^2\) The zeros on the right-hand sides of these equations result from the fact that the lateral load does not do work when $u$ or $v$ varies.
tions of the third degree involving the parameters \( a_{mn} \) alone. These equations can then be solved numerically in each particular case by successive approximations.

Numerical values of all the parameters have been computed for various intensities of the load \( q \) and for three different shapes of the plate \( b/a = 1, b/a = 3/4, \) and \( b/a = 1/2 \) by assuming \( \nu = 0.3. \)

It can be seen from the expression for \( w \) that, if we know the constant \( a_{oo} \), we can at once obtain the deflection of the plate at the center. These deflections are graphically represented in Fig. 208, in which \( w_{max}/h \) is plotted against \( qb^4/Dh. \) For comparison the figure also includes the straight lines which represent the deflections calculated by using the theory of small deflections. Also included is the curve for \( b/a = 0, \) which represents deflections of an infinitely long plate calculated as explained in Art. 3 (see page 13). It can be seen that the deflections of finite plates with \( b/a < \frac{3}{4} \) are very close to those obtained for an infinitely long plate.

Knowing the displacements as given by expressions (b), we can calculate the strain of the middle plane and the corresponding membrane stresses from Eqs. (b) of the preceding article. The bending stresses can then be found from Eqs. (101) and (102) for the bending and twisting moments. By adding the membrane and the bending stresses, we obtain the total stress. The maximum values of this stress are at the middle of the long sides of plates. They are given in graphical form in Fig. 209. For comparison, the figure also includes straight lines representing the stresses obtained by the theory of small deflections and a curve \( b/a = 0 \) representing the stresses for an infinitely long plate. It would seem reasonable to expect the total stress to be greater for \( b/a = 0 \) than for \( b/a = \frac{1}{2} \) for any value of load. We see that the curve for \( b/a = 0 \) falls below the curves for \( b/a = \frac{1}{2} \) and \( b/a = \frac{3}{4} \). This is probably a result of approximations in the energy solution which arise out of the use of a finite number of constants. It indicates that the calculated stresses are in error on the safe side, i.e., that they are too large. The error for \( b/a = \frac{1}{2} \) appears to be about 10 per cent.

The energy method can also be applied in the case of large deflections of simply supported rectangular plates. However, as may be seen from the foregoing discussion of the case of clamped edges, the application of this method requires a considerable amount of computation. To get an approximate solution for a simply

![Fig. 208](image-url)
supported rectangular plate, a simple method consisting of a combination of the known solutions given by the theory of small deflections and the membrane theory can be used.\footnote{This method is recommended by Föppl; see "Drang und Zwang," p. 345.} This method will now be illustrated by a simple example of a square plate. We assume that the load \( q \) can be resolved into two parts \( q_1 \) and \( q_2 \) in such a manner that part \( q_1 \) is balanced by the bending and shearing stresses calculated by the theory of small deflections, part \( q_2 \) being balanced by the membrane stresses. The deflection at the center as calculated for a square plate with sides \( 2a \) by the theory of small deflections is\footnote{The factor 0.730 is obtained by multiplying the number 0.00406, given in Table 8, by 16 and by \( 12(1 - \nu^2) \) = 11.25.}

\[
w_0 = 0.730 \frac{q_1 a^4}{Eh^3}
\]

From this we determine

\[
q_1 = \frac{w_0 Eh^3}{0.730a^4} \quad (e)
\]
Considering the plate as a membrane and using formula (250), we obtain

\[ w_0 = 0.802a \sqrt[3]{\frac{q_2 a}{E h}} \]

from which

\[ q_2 = \frac{w_0^2 E h}{0.516a^4} \]  

(f)

The deflection \( w_0 \) is now obtained from the equation

\[ q = q_1 + q_2 = \frac{w_0 E h^3}{0.730a^4} + \frac{w_0^2 E h}{0.516a^4} \]

which gives

\[ q = \frac{w_0 E h^3}{a^4} \left( 1.37 + 1.94 \frac{w_0^2}{h^2} \right) \]  

(252)

After the deflection \( w_0 \) has been calculated from this equation, the loads \( q_1 \) and \( q_2 \) are found from Eqs. (e) and (f), and the corresponding stresses are calculated by using for \( q_1 \) the small deflection theory (see Art. 30) and for \( q_2 \), Eq. (251). The total stress is then the sum of the stresses due to the loads \( q_1 \) and \( q_2 \).

Another approximate method of practical interest is based on consideration of the expression (248) for the strain energy due to the stretching of the middle surface of the plate. This expression can be put in the form

\[ V = \frac{E h}{2(1 - v^2)} \iint (e^2 - 2(1 - v)e_2) \, dx \, dy \]  

(g)

in which

\[ e = \varepsilon_x + \varepsilon_y \quad e_2 = \varepsilon_x \varepsilon_y - \frac{1}{2} \gamma_{xy}^2 \]

A similar expression can be written in polar coordinates, \( e_2 \) being, in case of axial symmetry, equal to \( \varepsilon_r \varepsilon_\theta \). The energy of bending must be added, of course, to the energy (g) in order to obtain the total strain energy of the plate. Yet an examination of exact solutions, such as described in Art. 98, leads to the conclusion that terms of the differential equations due to the presence of the term \( e_2 \) in expression (g) do not much influence the final result.

Starting from the hypothesis that the term containing \( \varepsilon_\theta \) actually can be neglected in comparison with \( e^2 \), we arrive at the differential equation of the bent plate

\[ \Delta \Delta w - \alpha^2 \Delta w = \frac{q}{D} \]  

(h)

in which the quantity

\[ \alpha^2 = \frac{12}{h^2} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] \]  

(i)

proves to be a constant. From Eqs. (h) of Art. 101 it follows that the dilatation \( e = \varepsilon_x + \varepsilon_y \) then also remains constant throughout the middle surface of the bent plate. The problem in question, simplified in this way, thus becomes akin to problems discussed in Chap. 12.

For a circular plate under symmetrical loading, Eq. (i) must be replaced by

\[
\alpha^2 = \frac{12}{h^2} \left[ \frac{du}{dr} + \frac{u}{r} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 \right]
\]  

(j)

In this latter case the constants of integration of Eq. (h) along with the constant \( \alpha \) allow us to fulfill all conditions prescribed on the boundary of the plate. However, for a more accurate calculation of the membrane stresses \( N_r, N_t \) from the deflections, the first of the equations (231) should be used in place of the relation (j).

The calculation of the membrane stresses in rectangular plates proves to be relatively more cumbersome. As a whole, however, the procedure still remains much simpler than the handling of the exact equations (245) and (246), and the numerical results, in cases discussed till now, prove to have an accuracy satisfactory for technical purposes. Nevertheless some reservation appears opportune in application of this method as long as the hypothesis providing its basis lacks a straight mechanical interpretation.

103. Large Deflections of Rectangular Plates with Simply Supported Edges. An exact solution\(^1\) of this problem, treated in the previous article approximately, can be established by starting from the simultaneous equations (245) and (246).

The deflection of the plate (Fig. 59) may be taken in the Navier form

\[
w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]  

(a)

the boundary conditions with regard to the deflections and the bending moments thus being satisfied by any, yet unknown, values of the coefficients \( w_{mn} \). The given lateral pressure may be expanded in a double Fourier series

\[
q = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]  

(b)

A suitable expression for the Airy stress function, then, is

\[
F = \frac{P_x y^2}{2bh} + \frac{P_y x^2}{2ah} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}
\]  

(c)

where \( P_x \) and \( P_y \) denote the total tension load applied on the sides \( x = 0, a \) and \( y = 0, b \), respectively. Substituting the expressions (a) and (c) into Eq. (245), we arrive at the following relation between the coefficients of both series:

\[
f_{mn} = \frac{E}{4(m^2b/a + n^2a/b)^2} \sum b_{mn} w_n w_m
\]  

(d)

The sum includes all products for which \( r \pm p = m \) and \( s \pm q = n \). The coefficients \( b_{repq} \) are given by the expression
\[
b_{repq} = 2rspq \pm (r^2q^2 + s^2p^2)
\]
where the sign is positive for \( r + p = m \) and \( s - q = n \) or for \( r - p = m \) and \( s + q = n \), and is negative otherwise. Taking, for example, a square plate \((a = b)\), we obtain
\[
f_{z,t} = \frac{E}{1,600} (-4w_{1,1}w_{1,1} + 36w_{1,1}w_{2,3} + 36w_{1,1}w_{1,5} + 64w_{1,2}w_{1,6} \cdots)
\]
It still remains to establish a relation between the deflections, the stress function, and the lateral loading. Inserting expressions \((a)\), \((b)\), and \((c)\) into Eq. (246), we arrive at the equation
\[
q_{mn} = Dw_{mn} \pi^4 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 + P_xw_{mn} \frac{m^2\pi^2}{a^2b} + P_yw_{mn} \frac{n^2\pi^2}{ab^2} + \frac{h\pi^4}{4a^2b^2} \sum C_{repq}f_{z}w_{pq}
\]
The summation includes, this time, all products for which \( r \pm p = m \) and \( s \pm q = n \), and the coefficients are given by
\[
C_{repq} = \pm (rq \pm sp)^2 \quad \text{if } r \neq 0 \text{ and } s \neq 0
\]
and are twice this value otherwise. The first sign is positive if either \( r - p = m \) or \( s - q = n \) (but not simultaneously), and is negative in all other cases. The second sign is positive if \( r + p = m \) and \( s - q = n \) or \( r - p = m \) and \( s + q = n \), and is negative otherwise. For example,
\[
q_{1,3} = Dw_{1,3} \pi^4 \left( \frac{1}{a^2} + \frac{9}{b^2} \right)^2 + P_xw_{1,3} \frac{\pi^2}{a^2b} + P_yw_{1,3} \frac{9\pi^2}{ab^2} + \frac{h\pi^4}{4a^2b^2} (-8f_{0,2}w_{1,1} - 8f_{0,2}w_{1,5} + 100f_{1,4}w_{2,1} - 64f_{2,2}w_{3,1} + \cdots)
\]
In accordance with conditions occurring in airplane structures the plate is considered rigidly framed, all edges thus remaining straight \(^1\) after deformation. Then the elongation of the plate, say in the direction \( x \), is independent of \( y \). By Eqs. \((b)\) and \((f)\) of Art. 101 its value is equal to
\[
\delta_x = \int_0^a \frac{\partial u}{\partial x} \, dx = \int_0^a \left[ \frac{1}{E} \left( \frac{\partial^2 F}{\partial y^2} - \nu \frac{\partial^2 F}{\partial x^2} \right) - \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] \, dx
\]
Using the series \((a)\) and \((c)\), this yields
\[
\delta_x = \frac{P_xa}{bhE} - \frac{\nu P_y}{hE} - \frac{\pi^2}{8a} \sum_{m=1}^\infty \sum_{n=1}^\infty m^2w_{mn}^2
\]
i.e., an expression which in fact does not include \( y \). Similarly, one obtains
\[
\delta_y = \frac{P_yb}{ahE} - \frac{\nu P_x}{hE} - \frac{\pi^2}{8b} \sum_{m=1}^\infty \sum_{n=1}^\infty n^2w_{mn}^2
\]
\(^1\) A solution due to Kaiser, loc. cit., is free from this restriction.
Membrane stresses immovable edges, \( \nu = 0.316 \)

\[ \frac{\sigma x^2}{Eh^2} \]

Fig. 210

Extreme-fiber bending stresses immovable edges \( \nu = 0.316 \)

\[ \frac{(\sigma'_{x})_0 = (\sigma'_{y})_0}{Eh^2} \]

\[ (\sigma'_{x})_B = (\sigma'_{y})_C \]

\[ (\sigma'_{x})_A = (\sigma'_{y})_A \]

Fig. 211
With regard to the boundary conditions we again consider two cases:

1. All edges are immovable. Then $\delta_z = \delta_y = 0$ and Eqs. (i) and (j) allow us to express $P_z$ and $P_y$ through the coefficients $w_{mn}$.

2. The external edge load is zero in the plane of the plate. We have then simply $P_z = P_y = 0$.

Next we have to keep a limited number of terms in the series (a) and (b) and to substitute the corresponding expressions (d) in Eq. (f). Thus we obtain for any assumed number of the unknown coefficients $w_{mn}$ as many cubic equations. Having resolved these equations we calculate the coefficients (d) and are able to obtain all data regarding the stress and strain of the plate from the series (a) and (c). The accuracy of the solution can be judged by observing the change in the numerical results as the number of the coefficients $w_{mn}$ introduced in the calculation is gradually increased. Some data for the flexural and membrane stresses obtained in this manner in the case of a uniformly loaded square plate with immovable edges are given in Figs. 210 and 211.