Calculating the values of functions ψ_1, ψ_2 and their derivatives from the asymptotic formulas (296) and substituting the resulting values in Eqs. (j'), we obtain

$$C_1 = -269\,\frac{\gamma a^2}{E\alpha}\,\frac{1}{\sqrt{d + x_0}}\,N$$

$$C_2 = -299\,\frac{\gamma a^2}{E\alpha}\,\frac{1}{\sqrt{d + x_0}}\,N$$

where
$$N = (e^{-\xi/\sqrt{2}}\,\sqrt{2\pi\xi})_{\xi=21.45}$$

Substituting these values of the constants in expression (g') we find for the bending moment at the bottom

$$M_0 = 13{,}900 \text{ lb-in. per in.}$$

In the same manner, by using expression (h'), we find the magnitude of the shearing force at the bottom of the tank as

$$Q_0 = 527 \text{ lb per in.}$$

These results do not differ much from the values obtained earlier for a tank with uniform wall thickness (page 487).

119. Thermal Stresses in Cylindrical Shells. *Uniform Temperature Distribution.* If a cylindrical shell with free edges undergoes a uniform temperature change, no thermal stresses will be produced. But if the edges are supported or clamped, free expansion of the shell is prevented, and local bending stresses are set up at the edges. Knowing the thermal expansion of a shell when the edges are free, the values of the reactive moments and forces at the edges for any kind of symmetrical support can be readily obtained by using Eqs. (279) and (280), as was done in the cases shown in Fig. 241.

Temperature Gradient in the Radial Direction. Assume that t_1 and t_2 are the uniform temperatures of the cylindrical wall at the inside and the outside surfaces, respectively, and that the variation of the temperature through the thickness is linear. In such a case, at points at a large distance from the ends of the shell, there will be no bending, and the stresses can be calculated by using Eq. (51), which was derived for clamped plates (see page 50). Thus the stresses at the outer and the inner surfaces are

$$\sigma_x = \sigma_\varphi = \pm\,\frac{E\alpha(t_1 - t_2)}{2(1 - \nu)} \tag{a}$$

where the upper sign refers to the outer surface, indicating that a tensile stress will act on this surface if $t_1 > t_2$.

Near the ends there will usually be some bending of the shell, and the total thermal stresses will be obtained by superposing upon (a) such stresses as are necessary to satisfy the boundary conditions. Let us consider, as an example, the condition of free edges, in which case the stresses σ_x must vanish at the ends. In calculating the stresses and deformations in this case we observe that at the edge the stresses (a) result in uniformly distributed moments M_0 (Fig. 250a) of the amount

$$M_0 = -\frac{E\alpha(t_1 - t_2)h^2}{12(1 - \nu)} \tag{b}$$

To obtain a free edge, moments of the same magnitude but opposite in direction must be superposed (Fig. 250b). Hence the stresses at a free edge are obtained by superposing upon the stresses (a) the stresses produced by the moments $-M_0$ (Fig. 250b). These latter stresses can be readily calculated by using solution (278). From this solution it follows that

$$(M_x)_{x=0} = \frac{E\alpha(t_1 - t_2)h^2}{12(1 - \nu)} \qquad (M_\varphi)_{x=0} = \nu(M_x)_{x=0} = \frac{\nu E\alpha(t_1 - t_2)h^2}{12(1 - \nu)} \tag{c}$$

$$(N_\varphi)_{x=0} = -\frac{Eh}{a}(w)_{x=0} = \frac{Eh}{a}\frac{M_0}{2\beta^2 D} = \frac{Eh\alpha(t_1 - t_2)}{2\sqrt{3}(1 - \nu)}\sqrt{1 - \nu^2} \tag{d}$$

It is seen that at the free edge the maximum thermal stress acts in the circumferential direction and is obtained by adding to the stress (a) the stresses produced by the moment M_φ and the force N_φ. Assuming that $t_1 > t_2$, we thus obtain

$$(\sigma_\varphi)_{\max} = \frac{E\alpha(t_1 - t_2)}{2(1 - \nu)}\left(1 - \nu + \frac{\sqrt{1 - \nu^2}}{\sqrt{3}}\right) \tag{e}$$

For $\nu = 0.3$ this stress is about 25 per cent greater than the stress (a) calculated at points at a large distance from the ends. We can therefore conclude that if a crack will occur in a brittle material such as glass due to a temperature difference $t_1 - t_2$, it will start at the edge and will proceed in the axial direction. In a similar manner the stresses can also be calculated in cases in which the edges are clamped or supported.[1]

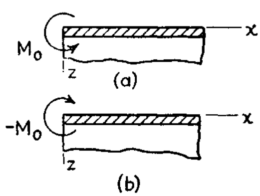

FIG. 250

Temperature Gradient in the Axial Direction. If the temperature is constant through the thickness of the wall but varies along the length of the cylinder, the problem can be easily reduced to the solution of Eq. (274).[2] Let $t = F(x)$ be the increase of the temperature of the shell from a certain uniform initial temperature. Assuming that the shell is divided into infinitely thin rings by planes perpendicular to the x axis and denoting the radius of the shell by a, the radial expansion of the rings due to the temperature change is $\alpha a F(x)$.

[1] Several examples of this kind are discussed in the paper by C. H. Kent, *Trans. ASME*, vol. 53, p. 167, 1931.

[2] See Timoshenko and Lessells, "Applied Elasticity," p. 146, 1925.

This expansion can be eliminated and the shell can be brought to its initial diameter by applying an external pressure of an intensity Z such that

$$\frac{a^2Z}{Eh} = \alpha a F(x)$$

which gives

$$Z = \frac{Eh\alpha}{a} F(x) \tag{f}$$

A load of this intensity entirely arrests the thermal expansion of the shell and produces in it only circumferential stresses having a magnitude

$$\sigma_\varphi = -\frac{aZ}{h} = -E\alpha F(x) \tag{g}$$

To obtain the total thermal stresses, we must superpose on the stresses (g) the stresses that will be produced in the shell by a load of the intensity $-Z$. This latter load must be applied in order to make the lateral surface of the shell free from the external load given by Eq. (f). The stresses produced in the shell by the load $-Z$ are obtained by the integration of the differential equation (276), which in this case becomes

$$\frac{d^4w}{dx^4} + 4\beta^4 w = -\frac{Eh\alpha}{Da} F(x) \tag{h}$$

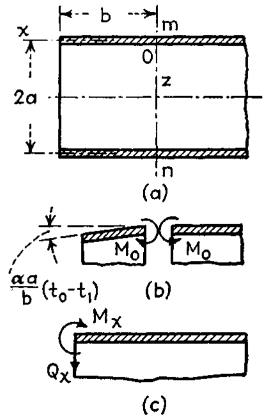

FIG. 251

As an example of the application of this equation let us consider a long cylinder, as shown in Fig. 251a, and assume that the part of the cylinder to the right of the cross section mn has a constant temperature t_0, whereas that to the left side has a temperature that decreases linearly to a temperature t_1 at the end $x = b$ according to the relation

$$t = t_0 - \frac{(t_0 - t_1)x}{b}$$

The temperature change at a point in this portion is thus

$$F(x) = t - t_0 = -\frac{(t_0 - t_1)x}{b} \tag{i}$$

Substituting this expression for the temperature change in Eq. (h), we find that the particular solution of that equation is

$$w_1 = \frac{\alpha a}{b}(t_0 - t_1)x \tag{j}$$

The displacement corresponding to this particular solution is shown in Fig. 251b, which indicates that there is at the section mn an angle of discontinuity of the magnitude

$$\frac{w_1}{x} = \frac{\alpha a}{b}(t_0 - t_1) \tag{k}$$

To remove this discontinuity the moments M_0 must be applied. Since the stress σ_φ corresponding to the particular solution (j) cancels the stresses (g), we conclude that the stresses produced by the moments M_0 are the total thermal stresses resulting from the above-described decrease in temperature. If the distances of the cross section mn from the ends of the cylinder are large, the magnitude of the moment M_0 can be obtained at once from Eq. (280) by substituting

$$Q_0 = 0 \qquad \left(\frac{dw_1}{dx}\right)_{x=0} = -\frac{\alpha a}{2b}(t_0 - t_1)$$

to obtain[1]

$$M_0 = -\beta D \frac{\alpha a}{2b}(t_0 - t_1) \tag{l}$$

Substituting for β its value from expression (275) and taking $\nu = 0.3$, we find that the maximum thermal stress is

$$(\sigma_x)_{\max} = \frac{6M_0}{h^2} = 0.353\frac{E\alpha}{b}\sqrt{ah}(t_0 - t_1) \tag{m}$$

It was assumed in this calculation that the length b to the end of the cylinder is large. If this is not the case, a correction to the moment (l) must be calculated as follows. In an infinitely long shell the moment M_0 produces at the distance $x = b$ a moment and a shearing force (Fig. 251c)[2] that are given by the general solution (282) as

$$\begin{aligned} M_x &= -D\frac{d^2w}{dx^2} = M_0\varphi(\beta b) \\ Q_x &= -D\frac{d^3w}{dx^3} = -2\beta M_0\zeta(\beta b) \end{aligned} \tag{n}$$

Since at the distance $x = b$ we have a free edge, it is necessary to apply there a moment and a force of the magnitude

$$-M_x = -M_0\varphi(\beta b) \qquad -Q_x = 2\beta M_0\zeta(\beta b) \tag{o}$$

in order to eliminate the forces (n) (Fig. 251c).

[1] If $t_0 - t_1$ is positive, as was assumed in the derivation, M_0 is negative and thus has the direction shown in Fig. 251b.

[2] The directions M_x and Q_x shown in Fig. 251c are the positive directions if the x axis has the direction shown in Fig. 251a.

The moment produced by the forces (*o*) at the cross section *mn* gives the desired correction ΔM_0 which is to be applied to the moment (*l*). Its value can be obtained from the third of the equations (282) if we substitute in it $-M_0\varphi(\beta b)$ for M_0 and $-2\beta M_0\zeta(\beta b)$* for Q_0. These substitutions give

$$\Delta M = -D\frac{d^2w}{dx^2} = -M_0[\varphi(\beta b)]^2 - 2M_0[\zeta(\beta b)]^2 \tag{p}$$

As a numerical example, consider a cast-iron cylinder having the following dimensions: $a = 9\frac{11}{16}$ in., $h = 1\frac{3}{8}$ in., $b = 4\frac{1}{4}$ in.; $\alpha = 101 \cdot 10^{-7}$, $E = 14 \cdot 10^6$ psi,

$$t_0 - t_1 = 180°\text{C}$$

The formula (*m*) then gives

$$\sigma_{\max} = 7{,}720 \text{ psi} \tag{q}$$

In calculating the correction (*p*), we have

$$\beta = \sqrt[4]{\frac{3(1-\nu^2)}{a^2h^2}} = \frac{1}{2.84}\,(\text{in.})^{-1} \qquad \beta b = 1.50$$

and, from Table 84,

$$\varphi(\beta b) = 0.238 \qquad \zeta(\beta b) = 0.223$$

Hence, from Eq. (*p*),

$$\Delta M = -M_0(0.238^2 + 2 \cdot 0.223^2) = -0.156M_0$$

This indicates that the above-calculated maximum stress (*q*) must be diminished by 15.6 per cent to obtain the correct maximum value of the thermal stress.

The method shown here for the calculation of thermal stresses in the case of a linear temperature gradient (*i*) can also be easily applied in cases in which $F(x)$ has other than a linear form.

120. Inextensional Deformation of a Circular Cylindrical Shell.[1] If the ends of a thin circular cylindrical shell are free and the loading is not symmetrical with respect to the axis of the cylinder, the deformation consists principally in bending. In such cases the magnitude of deflection can be obtained with sufficient accuracy by neglecting entirely the strain in the middle surface of the shell. An example of such a loading condition is shown in Fig. 252. The shortening of the vertical diameter along which the forces P act can be found with good accuracy by considering only the bending of the shell and assuming that the middle surface is inextensible.

Let us first consider the limitations to which the components of displacement are subject if the deformation of a cylindrical shell is to be inextensional. Taking an element in the middle surface of the shell at a point O and directing the coordinate axes as shown in Fig. 253, we shall

* The opposite sign to that in expression (*o*) is used here, since Eqs. 282 are derived for the direction of the x axis opposite to that shown in Fig. 251*a*.

[1] The theory of inextensional deformations of shells is due to Lord Rayleigh, *Proc. London Math. Soc.*, vol. 13, 1881, and *Proc. Roy. Soc.* (*London*), vol. 45, 1889.

denote by u, v, and w the components in the x, y, and z directions of the displacement of the point O. The strain in the x direction is then

$$\epsilon_x = \frac{\partial u}{\partial x} \tag{a}$$

In calculating the strain in the circumferential direction we use Eq. (*a*) (Art. 108, page 446). Thus,

$$\epsilon_\varphi = \frac{1}{a}\frac{\partial v}{\partial \varphi} - \frac{w}{a} \tag{b}$$

The shearing strain in the middle surface can be expressed by

$$\gamma_{x\varphi} = \frac{\partial u}{a\,\partial \varphi} + \frac{\partial v}{\partial x} \tag{c}$$

which is the same as in the case of small deflections of plates except that $a\,d\varphi$ takes the place of dy. The condition that the deformation is inexten-

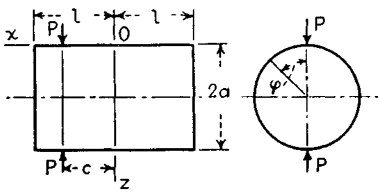

FIG. 252

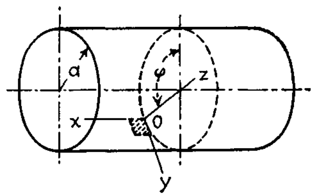

FIG. 253

sional then requires that the three strain components in the middle surface must vanish; *i.e.*,

$$\frac{\partial u}{\partial x} = 0 \qquad \frac{1}{a}\frac{\partial v}{\partial \varphi} - \frac{w}{a} = 0 \qquad \frac{\partial u}{a\,\partial \varphi} + \frac{\partial v}{\partial x} = 0 \tag{d}$$

These requirements are satisfied if we take the displacements in the following form:

$$\begin{aligned} u_1 &= 0 \\ v_1 &= a \sum_{n=1}^{\infty} (a_n \cos n\varphi - a_n' \sin n\varphi) \\ w_1 &= -a \sum_{n=1}^{\infty} n(a_n \sin n\varphi + a_n' \cos n\varphi) \end{aligned} \tag{e}$$

where a is the radius of the middle surface of the shell, φ the central angle, and a_n and a_n' constants that must be calculated for each particular case of loading. The displacements (*e*) represent the case in which all cross sections of the shell deform identically. On these displacements we can superpose displacements two of which vary along the length of the cylinder and which are given by the following series:

$$
\begin{aligned}
u_2 &= -a \sum_{n=1}^{\infty} \frac{1}{n} (b_n \sin n\varphi + b'_n \cos n\varphi) \\
v_2 &= x \sum_{n=1}^{\infty} (b_n \cos n\varphi - b'_n \sin n\varphi) \qquad (f) \\
w_2 &= -x \sum_{n=1}^{\infty} n(b_n \sin n\varphi + b'_n \cos n\varphi)
\end{aligned}
$$

It can be readily proved by substitution in Eqs. (*d*) that these expressions also satisfy the conditions of inextensibility. Thus the general expressions for displacements in inextensional deformation of a cylindrical shell are

$$u = u_1 + u_2 \qquad v = v_1 + v_2 \qquad w = w_1 + w_2 \qquad (g)$$

In calculating the inextensional deformations of a cylindrical shell under the action of a given system of forces, it is advantageous to use the energy method. To establish the required expression for the strain energy of bending of the shell, we begin with the calculation of the changes of curvature of the middle surface of the shell. The change of curvature in the direction of the generatrix is equal to zero, since, as can be seen from expressions (*e*) and (*f*), the generatrices remain straight. The change of curvature of the circumference is obtained by comparing the curvature of an element mn of the circumference (Fig. 254) before deformation with that of the corresponding element m_1n_1 after deformation. Before deformation the curvature in the circumferential direction is

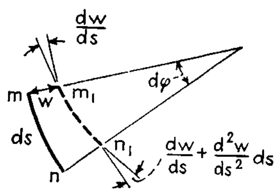

FIG. 254

$$\frac{\partial \varphi}{\partial s} = \frac{\partial \varphi}{a\, \partial \varphi} = \frac{1}{a}$$

The curvature of the element m_1n_1 after deformation is

$$\frac{\partial \varphi_1}{\partial s_1} \approx \frac{d\varphi + \dfrac{\partial^2 w}{\partial s^2}\, ds}{(a - w)\, d\varphi}$$

Hence the change in curvature is

$$\chi_\varphi = \frac{d\varphi + \frac{\partial^2 w}{\partial s^2}\,ds}{(a - w)\,d\varphi} - \frac{d\varphi}{a\,\partial\varphi} \approx \frac{1}{a^2}\left(w + \frac{\partial^2 w}{\partial \varphi^2}\right)$$

By using the second of the equations (d) we can also write

$$\chi_\varphi = \frac{1}{a^2}\left(\frac{\partial v}{\partial \varphi} + \frac{\partial^2 w}{\partial \varphi^2}\right) \tag{h}$$

The bending moment producing this change in curvature is

$$M_\varphi = -\frac{D}{a^2}\left(\frac{\partial v}{\partial \varphi} + \frac{\partial^2 w}{\partial \varphi^2}\right)$$

and the corresponding strain energy of bending per unit area can be calculated as in the discussion of plates (see page 46) and is equal to

$$\frac{D}{2a^4}\left(\frac{\partial v}{\partial \varphi} + \frac{\partial^2 w}{\partial \varphi^2}\right)^2 = \frac{D}{2a^4}\left(w + \frac{\partial^2 w}{\partial \varphi^2}\right)^2 \tag{i}$$

In addition to bending, there will be a twist of each element such as that shown at point O in Fig. 253. In calculating this twist we note that during deformation an element of a generatrix rotates[1] through an angle equal to $-\partial w/\partial x$ about the y axis and through an angle equal to $\partial v/\partial x$ about the z axis. Considering a similar element of a generatrix at a circumferential distance $a\,d\varphi$ from the first one, we see that its rotation about the y axis, as a result of the displacement w, is

$$-\frac{\partial w}{\partial x} - \frac{\partial^2 w}{\partial \varphi\,\partial x}\,d\varphi \tag{j}$$

The rotation of the same element in the plane tangent to the shell is

$$\frac{\partial v}{\partial x} + \frac{\partial\left(\frac{\partial v}{\partial x}\right)}{\partial \varphi}\,d\varphi$$

Because of the central angle $d\varphi$ between the two elements, the latter rotation has a component with respect to the y axis equal to[2]

$$-\frac{\partial v}{\partial x}\,d\varphi \tag{k}$$

From results (j) and (k) we conclude that the total angle of twist between the two elements under consideration is

$$-\chi_{x\varphi}a\,d\varphi = -\left(\frac{\partial^2 w}{\partial \varphi\,\partial x} + \frac{\partial v}{\partial x}\right)d\varphi$$

[1] In determining the sign of rotation the right-hand screw rule is used.
[2] A small quantity of second order is neglected in this expression.

and that the amount of strain energy per unit area due to twist is (see page 47)

$$\frac{D(1-\nu)}{a^2}\left(\frac{\partial^2 w}{\partial \varphi\, \partial x} + \frac{\partial v}{\partial x}\right)^2 \tag{l}$$

Adding together expressions (i) and (l) and integrating over the surface of the shell, the total strain energy of a cylindrical shell undergoing an inextensional deformation is found to be

$$V = \frac{D}{2a^4}\iint \left[\left(\frac{\partial v}{\partial \varphi} + \frac{\partial^2 w}{\partial \varphi^2}\right)^2 + 2(1-\nu)a^2\left(\frac{\partial^2 w}{\partial \varphi\, \partial x} + \frac{\partial v}{\partial x}\right)^2\right] a\, d\varphi\, dx$$

Substituting for w and v their expressions (g) and integrating, we find for a cylinder of a length $2l$ (Fig. 252) the following expression for strain energy:

$$V = \pi D l \sum_{n=2}^{\infty} \frac{(n^2-1)^2}{a^3}\left\{n^2\left[a^2(a_n^2 + a_n'^2) + \frac{1}{3}l^2(b_n^2 + b_n'^2)\right] + 2(1-\nu)a^2(b_n^2 + b_n'^2)\right\} \tag{297}$$

This expression does not contain a term with $n = 1$, since the corresponding displacements

$$\begin{aligned} v_1 &= a(a_1 \cos\varphi - a_1' \sin\varphi) \\ w_1 &= -a(a_1 \sin\varphi + a_1' \cos\varphi) \end{aligned} \tag{m}$$

represent the displacement of the circle in its plane as a rigid body. The vertical and horizontal components of this displacement are found by substituting $\varphi = \pi/2$ in expressions (m) to obtain

$$(v_1)_{\varphi=\pi/2} = -aa_1' \qquad (w_1)_{\varphi=\pi/2} = -aa_1$$

Such a displacement does not contribute to the strain energy.

The same conclusion can also be made regarding the displacements represented by the terms with $n = 1$ in expressions (f).

Let us now apply expression (297) for the strain energy to the calculation of the deformation produced in a cylindrical shell by two equal and opposite forces P acting along a diameter at a distance c from the middle[1] (Fig. 252). These forces produce work only during radial displacements w of their points of application, *i.e.*, at the points $x = c$, $\varphi = 0$, and $\varphi = \pi$. Also, since the terms with coefficients a_n and b_n in the expressions for w_1 and w_2 [see Eqs. (e) and (f)] vanish at these points, only terms with coefficients a_n' and b_n' will enter in the expression for deformation. By using the

[1] The case of a cylindrical shell reinforced by elastic rings with two opposite forces acting along a diameter of every ring was discussed by R. S. Levy, *J. Appl. Mechanics*, vol. 15, p. 30, 1948.

principle of virtual displacements, the equations for calculating the coefficients a_n' and b_n' are found to be

$$\frac{\partial V}{\partial a_n'}\,\delta a_n' = -na\,\delta a_n'(1 + \cos n\pi)P$$
$$\frac{\partial V}{\partial b_n'}\,\delta b_n' = -nc\,\delta b_n'(1 + \cos n\pi)P$$

Substituting expression (297) for V, we obtain, for the case where n is an even number,

$$\begin{aligned} a_n' &= -\frac{a^2P}{n(n^2-1)^2\pi Dl} \\ b_n' &= -\frac{ncPa^3}{(n^2-1)^2\pi Dl[\frac{1}{3}n^2l^2 + 2(1-\nu)a^2]} \end{aligned} \tag{n}$$

If n is an odd number, we obtain

$$a_n' = b_n' = 0 \tag{o}$$

Hence in this case, from expressions (e) and (f),

$$\begin{aligned} u &= \frac{Pa^3}{\pi Dl}\sum_{n=2,4,6,\ldots}\frac{ac\cos n\varphi}{(n^2-1)^2[\frac{1}{3}n^2l^2 + 2(1-\nu)a^2]} \\ v &= \frac{Pa^3}{\pi Dl}\sum_{n=2,4,6,\ldots}\left\{\frac{1}{n(n^2-1)^2} + \frac{ncx}{(n^2-1)^2[\frac{1}{3}n^2l^2 + 2(1-\nu)a^2]}\right\}\sin n\varphi \\ w &= \frac{Pa^3}{\pi Dl}\sum_{n=2,4,6,\ldots}\left\{\frac{1}{(n^2-1)^2} + \frac{n^2cx}{(n^2-1)^2[\frac{1}{3}n^2l^2 + 2(1-\nu)a^2]}\right\}\cos n\varphi \end{aligned} \tag{p}$$

If the forces P are applied at the middle, $c = 0$ and the shortening of the vertical diameter of the shell is

$$\delta = (w)_{\varphi=0} + (w)_{\varphi=\pi} = \frac{2Pa^3}{\pi Dl}\sum_{n=2,4,6,\ldots}\frac{1}{(n^2-1)^2} = 0.149\,\frac{Pa^3}{2Dl} \tag{q}$$

The increase in the horizontal diameter is

$$\delta_1 = -[(w)_{\varphi=\pi/2} + (w)_{\varphi=3\pi/2}] = \frac{2Pa^3}{\pi Dl}\sum_{n=2,4,6,\ldots}\frac{(-1)^{n/2+1}}{(n^2-1)^2} = 0.137\,\frac{Pa^3}{2Dl} \tag{r}$$

The change in length of any other diameter can also be readily calculated. The same calculations can also be made if c is different from zero, and the deflections vary with the distance x from the middle.

Solution (p) does not satisfy the conditions at the free edges of the shell, since it requires the distribution of moments $M_x = \nu M_\varphi$ to prevent any bending in meridional planes. This bending is, however, of a local character and does not substantially affect the deflections (q) and (r), which are in satisfactory agreement with experiments.

The method just described for analyzing the inextensional deformation of cylindrical shells can also be used in calculating the deformation of a portion of a cylindrical shell which is cut from a complete cylinder of radius a by two axial sections making an angle α with one another (Fig. 255). For example, taking for the displacements the series

$$
\begin{aligned}
u &= -\frac{\alpha a}{\pi} \sum \frac{b_n}{n} \sin \frac{n\pi\varphi}{\alpha} \\
v &= a \sum a_n \cos \frac{n\pi\varphi}{\alpha} + x \sum b_n \cos \frac{n\pi\varphi}{\alpha} \\
w &= -\frac{\pi a}{\alpha} \sum n a_n \sin \frac{n\pi\varphi}{\alpha} - \frac{x\pi}{\alpha} \sum n b_n \sin \frac{n\pi\varphi}{\alpha}
\end{aligned}
$$

we obtain an inextensional deformation of the shell such that the displacements u and w and also the bending moments M_φ vanish along the edges mn and m_1n_1. Such conditions are obtained if the shell is supported at points m, n, m_1, n_1 by bars directed radially and is loaded by a load P in the plane of symmetry. The deflection produced by this load can be found by applying the principle of virtual displacements.

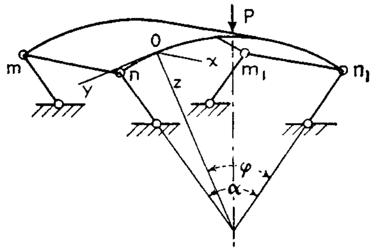

FIG. 255

121. General Case of Deformation of a Cylindrical Shell.[1]

To establish the differential equations for the displacements u, v, and w which define the deformation of a shell, we proceed as in the case of plates. We begin with the equations of equilibrium of an element cut out from the cylindrical shell by two adjacent axial sections and by two adjacent sections perpendicular to the axis of the cylinder (Fig. 253). The corresponding element of the middle surface of the shell after deformation is shown in Fig. 256a and b. In Fig. 256a the resultant forces and in Fig. 256b the

[1] A general theory of bending of thin shells has been developed by A. E. H. Love; see *Phil. Trans. Roy. Soc. (London)*, ser. A, p. 491, 1888; and his book "Elasticity," 4th ed., chap. 24, p. 515, 1927; see also H. Lamb, *Proc. London Math. Soc.*, vol. 21. For bending of cylindrical shells see also H. Reissner, *Z. angew. Math. Mech.*, vol. 13, p. 133, 1933; L. H. Donnell, *NACA Rept.* 479, 1933 (simplified theory); E. Torroja and J. Batanero, "Cubiertos laminares cilindros," Madrid, 1950; H. Parkus, *Österr. Ingr.-Arch.*, vol. 6, p. 30, 1951; W. Zerna, *Ingr.-Arch.*, vol. 20, p. 357, 1952; P. Csonka, *Acta Tech. Acad. Sci. Hung.*, vol. 6, p. 167, 1953. The effect of a concentrated load has been considered by A. Aas-Jakobsen, *Bauingenieur*, vol. 22, p. 343, 1941; by Y. N. Rabotnov, *Doklady Akad. Nauk S.S.S.R.*, vol. 3, 1946; and by V. Z. Vlasov, "A General Theory of Shells," Moscow, 1949. For cylindrical shells stiffened by ribs, see N. J. Hoff, *J. Appl. Mechanics*, vol. 11, p. 235, 1944; "H. Reissner Anniversary Volume," Ann Arbor, Mich., 1949; and W. Schnell, *Z. Flugwiss.*, vol. 3, p. 385, 1955. Anisotropic shells (together with a general theory) have been treated by W. Flügge, *Ingr.-Arch.*, vol. 3, p. 463, 1932; also by Vlasov, *op. cit.*, chaps. 11 and 12. For stress distribution around stiffened cutouts, see bibliography in L. S. D. Morley's paper, *Natl. Luchtvaarlab. Rappts.*, p. 362, Amsterdam, 1950. A theory of thick cylindrical shells is due to Z. Bazant, *Proc. Assoc. Bridge Structural Engrs.*, vol. 4, 1936.

resultant moments, discussed in Art. 104, are shown. Before deformation, the axes x, y, and z at any point O of the middle surface had the directions of the generatrix, the tangent to the circumference, and the normal to the middle surface of the shell, respectively. After deformation, which is assumed to be very small, these directions are slightly changed. We then take the z axis normal to the deformed middle surface, the x axis in the direction of a tangent to the generatrix, which may have become curved, and the y axis perpendicular to the xz plane. The

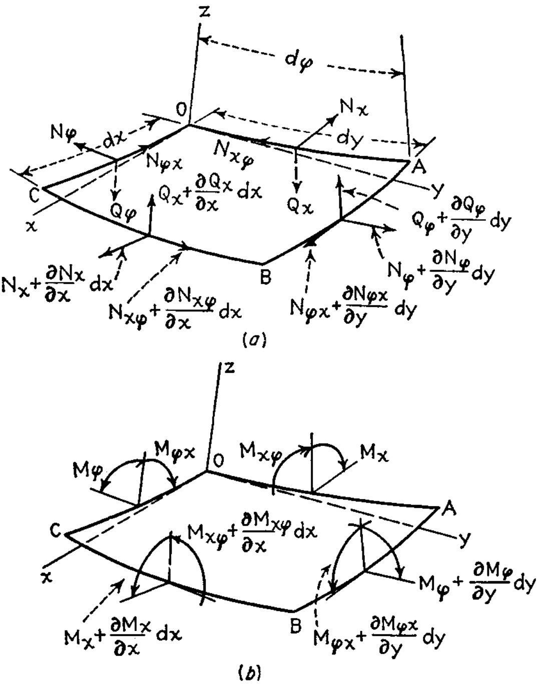

FIG. 256

directions of the resultant forces will also have been slightly changed accordingly, and these changes must be considered in writing the equations of equilibrium of the element $OABC$.

Let us begin by establishing formulas for the angular displacements of the sides BC and AB with reference to the sides OA and OC of the element, respectively. In these calculations we consider the displacements u, v, and w as very small, calculate the angular motions produced by each of these displacements, and obtain the resultant angular displacement by superposition. We begin with the rotation of the side BC with respect to the side OA. This rotation can be resolved into three component rotations with respect to the x, y, and z axes. The rotations of the sides OA and BC with respect to the x axis are caused by the displacements v and w. Since the displacements v represent motion of the sides OA and BC in the circumferential direction (Fig. 253), if a is the radius of the middle surface of the cylinder, the corresponding rotation of side OA about the x axis is v/a, and that of side BC is

$$\frac{1}{a}\left(v + \frac{\partial v}{\partial x}\,dx\right)$$

Thus, owing to the displacements v, the relative angular motion of BC with respect to OA about the x axis is

$$\frac{1}{a}\frac{\partial v}{\partial x}\,dx \tag{a}$$

Because of the displacements w, the side OA rotates about the x axis through the angle $\partial w/(a\,d\varphi)$, and the side BC through the angle

$$\frac{\partial w}{a\,\partial\varphi} + \frac{\partial}{\partial x}\left(\frac{\partial w}{a\,\partial\varphi}\right)dx$$

Thus, because of the displacements w, the relative angular displacement is

$$\frac{\partial}{\partial x}\left(\frac{\partial w}{a\,\partial\varphi}\right)dx \tag{b}$$

Summing up (a) and (b), the relative angular displacement about the x axis of side BC with respect to side OA is

$$\frac{1}{a}\left(\frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x\,\partial\varphi}\right)dx \tag{c}$$

The rotation about the y axis of side BC with respect to side OA is caused by bending of the generatrices in axial planes and is equal to

$$-\frac{\partial^2 w}{\partial x^2}\,dx \tag{d}$$

The rotation about the z axis of side BC with respect to side OA is due to bending of the generatrices in tangential planes and is equal to

$$\frac{\partial^2 v}{\partial x^2}\,dx \tag{e}$$

The formulas (c), (d), and (e) thus give the three components of rotation of the side BC with respect to the side OA.

Let us now establish the corresponding formulas for the angular displacement of side AB with respect to side OC. Because of the curvature of the cylindrical shell, the initial angle between these lateral sides of the element $OABC$ is $d\varphi$. However, because of the displacements v and w this angle will be changed. The rotation of the lateral side OC about the x axis is

$$\frac{v}{a} + \frac{\partial w}{a\,d\varphi} \tag{f}$$

The corresponding rotation for the lateral side AB is

$$\frac{v}{a} + \frac{\partial w}{a\,d\varphi} + \frac{d}{d\varphi}\left(\frac{v}{a} + \frac{\partial w}{a\,d\varphi}\right) d\varphi$$

Hence, instead of the initial angle $d\varphi$, we must now use the expression

$$d\varphi + d\varphi\left(\frac{\partial v}{a\,\partial\varphi} + \frac{\partial^2 w}{a\,\partial\varphi^2}\right) \tag{g}$$

In calculating the angle of rotation about the y axis of side AB with respect to the side OC we use the expression for twist from the preceding article (see page 504); this gives the required angular displacement as

$$-\left(\frac{\partial^2 w}{\partial\varphi\,\partial x} + \frac{\partial v}{\partial x}\right) d\varphi \tag{h}$$

Rotation about the z axis of the side AB with respect to OC is caused by the displacements v and w. Because of the displacement v, the angle of rotation of side OC is $\partial v/\partial x$, and that of side AB is

$$\frac{\partial v}{\partial x} + \frac{\partial}{a\,\partial\varphi}\left(\frac{\partial v}{\partial x}\right) a\,d\varphi$$

so that the relative angular displacement is

$$\frac{\partial}{a\,\partial\varphi}\left(\frac{\partial v}{\partial x}\right) a\,d\varphi \tag{i}$$

Because of the displacement w, the side AB rotates in the axial plane by the angle $\partial w/\partial x$. The component of this rotation with respect to the z axis is

$$-\frac{\partial w}{\partial x}\,d\varphi \tag{j}$$

Summing up (i) and (j), the relative angular displacement about the z axis of side AB with respect to side OC is

$$\left(\frac{\partial^2 v}{\partial\varphi\,\partial x} - \frac{\partial w}{\partial x}\right) d\varphi \tag{k}$$

Having the foregoing formulas[1] for the angles, we may now obtain three equations of equilibrium of the element $OABC$ (Fig. 256) by projecting all forces on the x, y, and z axes. Beginning with those forces

[1] These formulas can be readily obtained for a cylindrical shell from the general formulas given by A. E. H. Love in his book "Elasticity," 4th ed., p. 523, 1927.

parallel to the resultant forces N_x and $N_{\varphi x}$ and projecting them on the x axis, we obtain

$$\frac{\partial N_x}{\partial x}\,dx\,a\,d\varphi \qquad \frac{\partial N_{\varphi x}}{\partial \varphi}\,d\varphi\,dx$$

Because of the angle of rotation represented by expression (k), the forces parallel to N_y give a component in the x direction equal to

$$-N_\varphi\left(\frac{\partial^2 v}{\partial\varphi\,\partial x}-\frac{\partial w}{\partial x}\right)d\varphi\,dx$$

Because of the rotation represented by expression (e), the forces parallel to $N_{x\varphi}$ give a component in the x direction equal to

$$-N_{x\varphi}\frac{\partial^2 v}{\partial x^2}\,dx\,a\,d\varphi$$

Finally, because of angles represented by expressions (d) and (h), the forces parallel to Q_x and Q_φ give components in the x direction equal to

$$-Q_x\frac{\partial^2 w}{\partial x^2}\,dx\,a\,d\varphi - Q_\varphi\left(\frac{\partial^2 w}{\partial\varphi\,\partial x}+\frac{\partial v}{\partial x}\right)d\varphi\,dx$$

Regarding the external forces acting on the element, we assume that there is only a normal pressure of intensity q, the projection of which on the x and y axes is zero.

Summing up all the projections calculated above, we obtain

$$\begin{aligned}\frac{\partial N_x}{\partial x}\,dx\,a\,d\varphi + \frac{\partial N_{\varphi x}}{\partial \varphi}\,d\varphi\,dx - N_\varphi\left(\frac{\partial^2 v}{\partial\varphi\,\partial x}-\frac{\partial w}{\partial x}\right)d\varphi\,dx \\ - N_{x\varphi}\frac{\partial^2 v}{\partial x^2}\,dx\,a\,d\varphi - Q_x\frac{\partial^2 w}{\partial x^2}\,dx\,a\,d\varphi - Q_\varphi\left(\frac{\partial^2 w}{\partial\varphi\,\partial x}+\frac{\partial v}{\partial x}\right)d\varphi\,dx = 0\end{aligned}$$

In the same manner two other equations of equilibrium can be written. After simplification, all three equations can be put in the following form:

$$\begin{aligned}&a\frac{\partial N_x}{\partial x}+\frac{\partial N_{\varphi x}}{\partial\varphi}-aQ_x\frac{\partial^2 w}{\partial x^2}-aN_{x\varphi}\frac{\partial^2 v}{\partial x^2}\\&\qquad -Q_\varphi\left(\frac{\partial v}{\partial x}+\frac{\partial^2 w}{\partial x\,\partial\varphi}\right)-N_\varphi\left(\frac{\partial^2 v}{\partial x\,\partial\varphi}-\frac{\partial w}{\partial x}\right)=0\\&\frac{\partial N_\varphi}{\partial\varphi}+a\frac{\partial N_{x\varphi}}{\partial x}+aN_x\frac{\partial^2 v}{\partial x^2}-Q_x\left(\frac{\partial v}{\partial x}+\frac{\partial^2 w}{\partial x\,\partial\varphi}\right)\\&\qquad +N_{\varphi x}\left(\frac{\partial^2 v}{\partial x\,\partial\varphi}-\frac{\partial w}{\partial x}\right)-Q_\varphi\left(1+\frac{\partial v}{a\,\partial\varphi}+\frac{\partial^2 w}{a\,\partial\varphi^2}\right)=0\\&a\frac{\partial Q_x}{\partial x}+\frac{\partial Q_\varphi}{\partial\varphi}+N_{x\varphi}\left(\frac{\partial v}{\partial x}+\frac{\partial^2 w}{\partial x\,\partial\varphi}\right)+aN_x\frac{\partial^2 w}{\partial x^2}\\&\qquad +N_\varphi\left(1+\frac{\partial v}{a\,\partial\varphi}+\frac{\partial^2 w}{a\,\partial\varphi^2}\right)+N_{\varphi x}\left(\frac{\partial v}{\partial x}+\frac{\partial^2 w}{\partial x\,\partial\varphi}\right)+qa=0\end{aligned} \tag{298}$$

Going now to the three equations of moments with respect to the x, y, and z axes (Fig. 256b) and again taking into consideration the small angular displacements of the sides BC and AB with respect to OA and OC, respectively, we obtain the following equations:

$$
\begin{aligned}
&a\frac{\partial M_{x\varphi}}{\partial x} - \frac{\partial M_\varphi}{\partial \varphi} - aM_x\frac{\partial^2 v}{\partial x^2} - M_{\varphi x}\left(\frac{\partial^2 v}{\partial x\,\partial\varphi} - \frac{\partial w}{\partial x}\right) + aQ_\varphi = 0\\
&\frac{\partial M_{\varphi x}}{\partial \varphi} + a\frac{\partial M_x}{\partial x} + aM_{x\varphi}\frac{\partial^2 v}{\partial x^2} - M_\varphi\left(\frac{\partial^2 v}{\partial x\,\partial\varphi} - \frac{\partial w}{\partial x}\right) - aQ_x = 0\\
&M_x\left(\frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x\,\partial\varphi}\right) + aM_{x\varphi}\frac{\partial^2 w}{\partial x^2} + M_{\varphi x}\left(1 + \frac{\partial v}{a\,\partial\varphi} + \frac{\partial^2 w}{a\,\partial\varphi^2}\right)\\
&\qquad - M_\varphi\left(\frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x\,\partial\varphi}\right) + a(N_{x\varphi} - N_{\varphi x}) = 0
\end{aligned}
\tag{299}
$$

By using the first two of these equations[1] we can eliminate Q_x and Q_φ from Eqs. (298) and obtain in this way three equations containing the resultant forces N_x, N_φ, and $N_{x\varphi}$ and the moments M_x, M_φ, and $M_{x\varphi}$. By using formulas (253) and (254) of Art. 104, all these quantities can be expressed in terms of the three strain components ϵ_x, ϵ_φ, and $\gamma_{x\varphi}$ of the middle surface and the three curvature changes χ_x, χ_φ, and $\chi_{x\varphi}$. By using the results of the previous article, these latter quantities can be represented in terms of the displacements u, v, and w as follows:[2]

$$
\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} & \epsilon_\varphi &= \frac{\partial v}{a\,\partial\varphi} - \frac{w}{a} & \gamma_{x\varphi} &= \frac{\partial u}{a\,\partial\varphi} + \frac{\partial v}{\partial x}\\
\chi_x &= \frac{\partial^2 w}{\partial x^2} & \chi_\varphi &= \frac{1}{a^2}\left(\frac{\partial v}{\partial\varphi} + \frac{\partial^2 w}{\partial\varphi^2}\right) & \chi_{x\varphi} &= \frac{1}{a}\left(\frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x\,\partial\varphi}\right)
\end{aligned}
\tag{300}
$$

Thus we finally obtain the three differential equations for the determination of the displacements u, v, and w.

In the derivation equations (298) and (299) the change of curvature of the element $OABC$ was taken into consideration. This procedure is necessary if the forces N_x, N_y, and N_{xy} are not small in comparison with their *critical* values, at which lateral buckling of the shell may occur.[3] If these forces are small, their effect on bending is negligible, and we can omit from Eqs. (298) and (299) all terms containing the products of the resultant forces or resultant moments with the derivatives of the small displacements u, v, and w. In such a case the three Eqs. (298) and the

[1] To satisfy the third of these equations the trapezoidal form of the sides of the element $OABC$ must be considered as mentioned in Art. 104. This question is discussed by W. Flügge, "Statik und Dynamik der Schalen," 2d ed., p. 148, Berlin, 1957.

[2] The same expressions for the change of curvature as in the preceding article are used, since the effect of strain in the middle surface on curvature is neglected.

[3] The problems of buckling of cylindrical shells are discussed in S. Timoshenko, "Theory of Elastic Stability," and will not be considered here.

first two equations of system (299) can be rewritten in the following simplified form:

$$
\begin{aligned}
a\frac{\partial N_x}{\partial x} + \frac{\partial N_{\varphi x}}{\partial \varphi} &= 0 \\
\frac{\partial N_\varphi}{\partial \varphi} + a\frac{\partial N_{x\varphi}}{\partial x} - Q_\varphi &= 0 \\
a\frac{\partial Q_x}{\partial x} + \frac{\partial Q_\varphi}{\partial \varphi} + N_\varphi + qa &= 0 \\
a\frac{\partial M_{x\varphi}}{\partial x} - \frac{\partial M_\varphi}{\partial \varphi} + aQ_\varphi &= 0 \\
\frac{\partial M_{\varphi x}}{\partial \varphi} + a\frac{\partial M_x}{\partial x} - aQ_x &= 0
\end{aligned}
\tag{301}
$$

Eliminating the shearing forces Q_x and Q_φ, we finally obtain the three following equations:

$$
\begin{aligned}
a\frac{\partial N_x}{\partial x} + \frac{\partial N_{\varphi x}}{\partial \varphi} &= 0 \\
\frac{\partial N_\varphi}{\partial \varphi} + a\frac{\partial N_{x\varphi}}{\partial x} + \frac{\partial M_{x\varphi}}{\partial x} - \frac{1}{a}\frac{\partial M_\varphi}{\partial \varphi} &= 0 \\
N_\varphi + \frac{\partial^2 M_{\varphi x}}{\partial x\,\partial \varphi} + a\frac{\partial^2 M_x}{\partial x^2} - \frac{\partial^2 M_{x\varphi}}{\partial x\,\partial \varphi} + \frac{1}{a}\frac{\partial^2 M_\varphi}{\partial \varphi^2} + qa &= 0
\end{aligned}
\tag{302}
$$

By using Eqs. (253), (254), and (300), all the quantities entering in these equations can be expressed by the displacements u, v, and w, and we obtain

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2a^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1+\nu}{2a}\frac{\partial^2 v}{\partial x\,\partial \varphi} - \frac{\nu}{a}\frac{\partial w}{\partial x} = 0 \\
&\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\,\partial \varphi} + a\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{a}\frac{\partial^2 v}{\partial \varphi^2} - \frac{1}{a}\frac{\partial w}{\partial \varphi} \\
&\qquad + \frac{h^2}{12a}\left(\frac{\partial^3 w}{\partial x^2\,\partial \varphi} + \frac{\partial^3 w}{a^2\,\partial \varphi^3}\right) + \frac{h^2}{12a}\left[(1-\nu)\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{a^2\,\partial \varphi^2}\right] = 0 \\
&\nu\frac{\partial u}{\partial x} + \frac{\partial v}{a\,\partial \varphi} - \frac{w}{a} - \frac{h^2}{12}\left(a\frac{\partial^4 w}{\partial x^4} + \frac{2}{a}\frac{\partial^4 w}{\partial x^2\,\partial \varphi^2} + \frac{\partial^4 w}{a^3\,\partial \varphi^4}\right) \\
&\qquad - \frac{h^2}{12}\left(\frac{2-\nu}{a}\frac{\partial^3 v}{\partial x^2\,\partial \varphi} + \frac{\partial^3 v}{a^3\,\partial \varphi^3}\right) = -\frac{aq(1-\nu^2)}{Eh}
\end{aligned}
\tag{303}
$$

More elaborate investigations show[1] that the last two terms on the left-hand side of the second of these equations and the last term on the left-hand side of the third equation are small quantities of the same order as those which we already disregarded by assuming a linear distribution of stress through the thickness of the shell and by neglecting the stretching of the middle surface of the shell (see page 431). In such a case it

[1] See Vlasov, *op. cit.*, p. 316, and, for more exact equations, p. 257.

will be logical to omit the above-mentioned terms and to use in analysis of thin cylindrical shells the following simplified system of equations:

$$\begin{gathered}
\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2a^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1+\nu}{2a}\frac{\partial^2 v}{\partial x\,\partial \varphi} - \frac{\nu}{a}\frac{\partial w}{\partial x} = 0 \\
\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\,\partial \varphi} + a\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{a}\frac{\partial^2 v}{\partial \varphi^2} - \frac{1}{a}\frac{\partial w}{\partial \varphi} = 0 \\
\nu\frac{\partial u}{\partial x} + \frac{\partial v}{a\,\partial \varphi} - \frac{w}{a} - \frac{h^2}{12}\left(a\frac{\partial^4 w}{\partial x^4} + \frac{2}{a}\frac{\partial^4 w}{\partial x^2\,\partial \varphi^2} + \frac{\partial^4 w}{a^3\,\partial \varphi^4}\right) = -\frac{aq(1-\nu^2)}{Eh}
\end{gathered} \tag{304}$$

Some simplified expressions for the stress resultants which are in accordance with the simplified relations (304) between the displacements of the shell will be given in Art. 125.

From the foregoing it is seen that the problem of a laterally loaded cylindrical shell reduces in each particular case to the solution of a system of three differential equations. Several applications of these equations will be shown in the next articles.

122. Cylindrical Shells with Supported Edges. Let us consider the case of a cylindrical shell supported at the ends and submitted to the

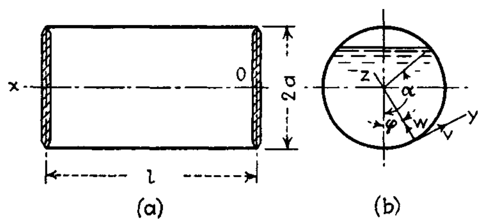

Fig. 257

pressure of an enclosed liquid as shown in Fig. 257.[1] The conditions at the supports and the conditions of symmetry of deformation will be satisfied if we take the components of displacement in the form of the following series:

$$\begin{aligned}
u &= \sum\sum A_{mn} \cos n\varphi \cos \frac{m\pi x}{l} \\
v &= \sum\sum B_{mn} \sin n\varphi \sin \frac{m\pi x}{l} \\
w &= \sum\sum C_{mn} \cos n\varphi \sin \frac{m\pi x}{l}
\end{aligned} \tag{a}$$

in which l is the length of the cylinder and φ is the angle measured as shown in Fig. 257.[2]

[1] See S. Timoshenko, "Theory of Elasticity," vol. 2, p. 385, St. Petersburg, 1916 (Russian).

[2] By substituting expressions (a) in Eqs. (300) it can be shown that the tensile forces N_x and the moments M_x vanish at the ends; the shearing forces do not vanish, however, since $\gamma_{x\varphi}$ and $M_{x\varphi}$ are not zero at the ends.

The intensity of the load q is represented by the following expressions:

$$\begin{aligned} q &= -\gamma a(\cos\varphi - \cos\alpha) && \text{when } \varphi < \alpha \\ q &= 0 && \text{when } \varphi > \alpha \end{aligned} \tag{b}$$

in which γ is the specific weight of the liquid and the angle α defines the level of the liquid, as shown in Fig. 257*b*. The load q can be represented by the series

$$q = \sum\sum D_{mn} \cos n\varphi \sin \frac{m\pi x}{l} \tag{c}$$

in which the coefficients D_{mn} can be readily calculated in the usual way from expressions (b). These coefficients are represented by the expression

$$D_{mn} = -\frac{8\gamma a}{mn\pi^2(n^2 - 1)} (\cos\alpha \sin n\alpha - n \cos n\alpha \sin\alpha) \tag{d}$$

where $m = 1, 3, 5, \ldots$ and $n = 2, 3, 4, \ldots$

whereas
$$D_{m0} = -\frac{4\gamma a}{m\pi^2} (\sin\alpha - \alpha\cos\alpha) \tag{e}$$

and
$$D_{m1} = -\frac{2\gamma a}{m\pi^2} (2\alpha - \sin 2\alpha) \tag{f}$$

In the case of a cylindrical shell completely filled with liquid, we denote the pressure at the axis of the cylinder[1] by γd; then

$$q = -\gamma(d + a\cos\varphi) \tag{g}$$

and we obtain, instead of expressions (d), (e), and (f),

$$D_{mn} = 0 \qquad D_{m0} = -\frac{4\gamma d}{m\pi} \qquad D_{m1} = -\frac{4\gamma a}{m\pi} \tag{h}$$

To obtain the deformation of the shell we substitute expressions (a) and (c) in Eqs. (304). In this way we obtain for each pair of values of m and n a system of three linear equations from which the corresponding values of the coefficients A_{mn}, B_{mn}, and C_{mn} can be calculated.[2] Taking a particular case in which $d = a$, we find that for $n = 0$ and $m = 1, 3, 5, \ldots$ these equations are especially simple, and we obtain

$$B_{m0} = 0 \qquad C_{m0} = -\frac{m\pi}{\lambda\nu} A_{m0} = -\frac{\pi N}{3m\left[\lambda^2(1 - \nu^2) + \frac{\eta^2}{3} m^4\pi^4\right]}$$

where
$$N = \frac{2\gamma a l^2 h}{\pi^2 D} \qquad \lambda = \frac{l}{a} \qquad \eta = \frac{h}{2l}$$

[1] In a closed cylindrical vessel this pressure can be larger than $a\gamma$.

[2] Such calculations have been made for several particular cases by I. A. Wojtaszak, *Phil. Mag.*, ser. 7, vol. 18, p. 1099, 1934; see also the paper by H. Reissner in *Z. angew. Math. Mech.*, vol. 13, p. 133, 1933.

For $n = 1$ the expressions for the coefficients are more complicated. To show how rapidly the coefficients diminish as m increases, we include in Table 87 the numerical values of the coefficients for a particular case in which $a = 50$ cm, $l = 25$ cm, $h = 7$ cm, $\nu = 0.3$, and $\alpha = \pi$.

TABLE 87. THE VALUES OF THE COEFFICIENTS IN EXPRESSIONS (*a*)

m	$A_{m0} \dfrac{2 \cdot 10^3}{Nh}$	$C_{m0} \dfrac{2 \cdot 10^3}{Nh}$	$A_{m1} \dfrac{2 \cdot 10^3}{Nh}$	$B_{m1} \dfrac{2 \cdot 10^3}{Nh}$	$C_{m1} \dfrac{2 \cdot 10^3}{Nh}$
1	57.88	−1,212.	49.18	−66.26	−1,183
3	0.1073	−6.742	0.1051	−0.0432	− 6.704
5	0.00503	−0.526	0.00499	−0.00122	− 0.525

It is seen that the coefficients rapidly diminish as m increases. Hence, by limiting the number of coefficients to those given in the table, we shall obtain the deformation of the shell with satisfactory accuracy.

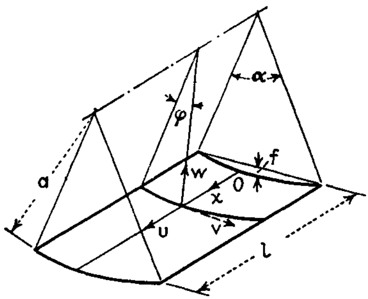

FIG. 258

123. Deflection of a Portion of a Cylindrical Shell. The method used in the preceding article can also be applied to a portion of a cylindrical shell which is supported along the edges and submitted to the action of a uniformly distributed load q normal to the surface (Fig. 258).[1] We take the components of displacement in the form of the series

$$
\begin{aligned}
u &= \sum\sum A_{mn} \sin \frac{n\pi\varphi}{\alpha} \cos \frac{m\pi x}{l} \\
v &= \sum\sum B_{mn} \cos \frac{n\pi\varphi}{\alpha} \sin \frac{m\pi x}{l} \qquad (a) \\
w &= \sum\sum C_{mn} \sin \frac{n\pi\varphi}{\alpha} \sin \frac{m\pi x}{l}
\end{aligned}
$$

in which α is the central angle subtended by the shell and l is the length of the shell. It can be shown by substitution of expressions (*a*) in Eqs. (300) that in this way we shall satisfy the conditions at the boundary, which require that along the edges $\varphi = 0$ and $\varphi = \alpha$ the deflection w, the force N_φ, and the moment M_φ vanish and that along the edges $x = 0$ and $x = l$ the deflection w, the force N_x, and the moment M_x vanish.

[1] See Timoshenko, "Theory of Elasticity," vol. 2, p. 386, 1916.

The intensity of the normal load q can be represented by the series

$$q = \sum\sum D_{mn} \sin \frac{n\pi\varphi}{\alpha} \sin \frac{m\pi x}{l} \tag{b}$$

Substituting series (a) and (b) in Eqs. (304), we obtain the following system of linear algebraic equations for calculating the coefficients A_{mn}, B_{mn}, and C_{mn}:

$$\begin{aligned}
A_{mn}\pi\left[\left(\frac{am}{l}\right)^2 + \frac{(1-\nu)n^2}{2\alpha^2}\right] + B_{mn}\pi\frac{(1+\nu)amn}{2\alpha l} + C_{mn}\frac{\nu am}{l} &= 0 \\
A_{mn}\pi\frac{(1+\nu)amn}{2\alpha l} + B_{mn}\pi\left[\frac{(1-\nu)a^2m^2}{2l^2} + \frac{n^2}{\alpha^2}\right] + C_{mn}\frac{n}{\alpha} &= 0 \\
A_{mn}\nu\pi\frac{am}{l} + B_{mn}\frac{n\pi}{\alpha} + C_{mn}\left[1 + \frac{\pi^4h^2}{12a^2}\left(\frac{a^2m^2}{l^2} + \frac{n^2}{\alpha^2}\right)^2\right] &= D_{mn}\frac{a^2(1-\nu^2)}{Eh}
\end{aligned} \tag{c}$$

To illustrate the application of these equations let us consider the case of a uniformly distributed load[1] acting on a portion of a cylindrical shell having a small angle α and a small sag $f = a[1 - \cos(\alpha/2)]$. In this particular case expression (b) becomes

$$q = \sum_{1,3,5,\ldots}\sum_{1,3,5,\ldots} \frac{16q}{\pi^2 mn} \sin\frac{m\pi x}{l} \sin\frac{n\pi\varphi}{\alpha} \tag{d}$$

and the coefficients D_{mn} are given by the expression

$$D_{mn} = \frac{16q}{mn\pi^2} \tag{e}$$

Substituting these values in Eqs. (c), we can calculate the coefficients A_{mn}, B_{mn}, and C_{mn}. The calculations made for a particular case in which $\alpha a = l$ and for several values of the ratio f/h show that for small values of this ratio, series (a) are rapidly convergent and the first few terms give the displacements with satisfactory accuracy.

The calculations also show that the maximum values of the bending stresses produced by the moments M_x and M_φ diminish rapidly as f/h increases. The calculation of these stresses is very tedious in the case of larger values of f/h, since the series representing the moments become less rapidly convergent and a larger number of terms must be taken.

The method used in this article is similar to Navier's method of calculating bending of rectangular plates with simply supported edges. If only the rectilinear edges $\varphi = 0$ and $\varphi = \alpha$ of the shell in Fig. 258 are simply supported and the other two edges are built in or free, a solution similar to that of M. Lévy's method for the case of rectangular plates (see page 113) can be applied. We assume the following series for the components of displacement:

[1] The load is assumed to act toward the axis of the cylinder.

$$
\begin{aligned}
u &= \sum U_m \sin \frac{m\pi\varphi}{\alpha} \\
v &= \sum V_m \cos \frac{m\pi\varphi}{\alpha} \\
w &= \sum W_m \sin \frac{m\pi\varphi}{\alpha}
\end{aligned}
\tag{f}
$$

in which U_m, V_m, and W_m are functions of x only. Substituting these series in Eqs. (304), we obtain for U_m, V_m, and W_m three ordinary differential equations with constant coefficients. These equations can be integrated by using exponential functions. An analysis of this kind made for a closed cylindrical shell[1] shows that the solution is very involved and that results suitable for practical application can be obtained only by introducing simplifying assumptions. It could be shown that each set of the functions U_m, V_m, W_m contains eight constants of integration for each assumed value of m. Accordingly, four conditions on each edge x = constant must be at our disposal. Let us formulate these conditions in the following three cases.

Built-in Edge. Usually such a support is considered as perfectly rigid, and the edge conditions then are

$$
u = 0 \qquad v = 0 \qquad w = 0 \qquad \frac{\partial w}{\partial x} = 0 \tag{g}
$$

Should it happen, however, that the shell surface on the edge is free to move in the direction x, then the first of the foregoing conditions has to be replaced by the condition $N_x = 0$.

Simply Supported Edge. Such a hinged edge is not able to transmit a moment M_x needed to enforce the condition $\partial w/\partial x = 0$. Assuming also that there is no edge resistance in the direction x, we arrive at the boundary conditions

$$
v = 0 \qquad w = 0 \qquad M_x = 0 \qquad N_x = 0 \tag{h}
$$

whereas the displacement u and the stress resultants $N_{x\varphi}$, $M_{x\varphi}$, and Q_x do not vanish on the edge.

The reactions of the simply supported edge (Fig. 259*a*) deserve brief consideration. The action of a twisting couple $M_{x\varphi}\,ds$, applied to an element $ABCD$ of the edge, is statically equivalent to the action of three forces shown in Fig. 259*b*. A variation of the radial forces $M_{x\varphi}$ along the edge yields, just as in the case of a plate (Fig. 50), an additional shearing force of the intensity $-\partial M_{x\varphi}/\partial s$, the total shearing force being (Fig. 259*c*)

$$
T_x = Q_x - \frac{\partial M_{x\varphi}}{a\,\partial\varphi} \tag{i}
$$

The remaining component $M_{x\varphi}\,d\varphi$ (Fig. 259*b*) may be considered as a supplementary membrane force of the intensity $M_{x\varphi}\,d\varphi/ds = M_{x\varphi}/a$. Hence the resultant membrane force in the direction of the tangent to the edge becomes

$$
S_x = N_{x\varphi} + \frac{M_{x\varphi}}{a} \tag{j}
$$

[1] See paper by K. Miesel, *Ingr.-Arch.*, vol. 1, p. 29, 1929. An application of the theory to the calculation of stress in the hull of a submarine is shown in this paper.

Free Edge. Letting all the stress resultants vanish on the edge, we find that the four conditions characterizing the free edge assume the form

$$N_x = 0 \qquad M_x = 0 \qquad S_x = 0 \qquad T_x = 0 \tag{k}$$

where S_x and T_x are given by expressions (j) and (i), respectively.[1]

124. An Approximate Investigation of the Bending of Cylindrical Shells. From the discussion of the preceding article it may be concluded that the application of the general theory of bending of cylindrical shells in even the simplest cases results in very complicated calculations. To make the theory applicable to the solution of practical problems some further simplifications in this theory are necessary. In considering the membrane theory of cylindrical shells it was stated that this theory gives satisfactory results for portions of a shell at a considerable distance from the edges but that it is insufficient to satisfy all the conditions at the boundary. It is logical, therefore, to take the solution furnished by the membrane theory as a first

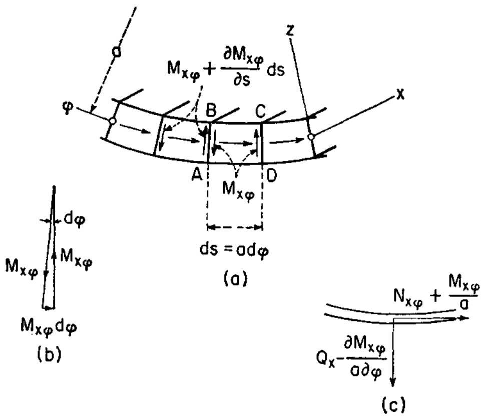

FIG. 259

approximation and use the more elaborate bending theory only to satisfy the conditions at the edges. In applying this latter theory, it must be assumed that no external load is distributed over the shell and that only forces and moments such as are necessary to satisfy the boundary conditions are applied along the edges. The bending produced by such forces can be investigated by using Eqs. (303) after placing the load q equal to zero in these equations.

In applications such as are encountered in structural engineering[2] the ends $x = 0$ and $x = l$ of the shell (Fig. 260) are usually supported in such a manner that the

[1] For a solution of the problem of bending based on L. H. Donnell's simplified differential equations see N. J. Hoff, *J. Appl. Mechanics*, vol. 21, p. 343, 1954; see also Art. 125 of this book.

[2] In recent times thin reinforced cylindrical shells of concrete have been successfully applied in structures as coverings for large halls. Descriptions of some of these structures can be found in the article by F. Dischinger, "Handbuch für Eisenbetonbau," 3d ed., vol. 12, Berlin, 1928; see also the paper by F. Dischinger and U. Finsterwalder in *Bauingenieur*, vol. 9, 1928, and references in Art. 126 of this book.

displacements v and w at the ends vanish. Experiments show that in such shells the bending in the axial planes is negligible, and we can assume $M_x = 0$ and $Q_x = 0$ in the equations of equilibrium (301). We can also neglect the twisting moment $M_{x\varphi}$. With these assumptions the system of Eqs. (301) can be considerably simplified, and

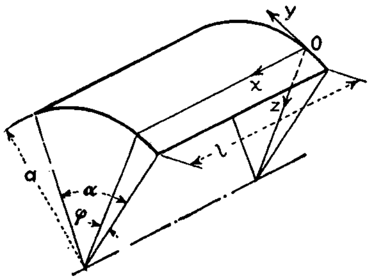

FIG. 260

the resultant forces and components of displacement can all be expressed in terms[1] of moment M_φ. From the fourth of the equations (301) we obtain

$$Q_\varphi = \frac{1}{a}\frac{\partial M_\varphi}{\partial \varphi} \tag{a}$$

Substituting this in the third equation of the same system, we obtain, for $q = 0$,

$$N_\varphi = -\frac{\partial Q_\varphi}{\partial \varphi} = -\frac{1}{a}\frac{\partial^2 M_\varphi}{\partial \varphi^2} \tag{b}$$

The second and the first of the equations (301) then give

$$\frac{\partial N_{x\varphi}}{\partial x} = \frac{1}{a}\left(Q_\varphi - \frac{\partial N_\varphi}{\partial \varphi}\right) = \frac{1}{a^2}\left(\frac{\partial M_\varphi}{\partial \varphi} + \frac{\partial^3 M_\varphi}{\partial \varphi^3}\right) \tag{c}$$

$$\frac{\partial^2 N_x}{\partial x^2} = -\frac{1}{a}\frac{\partial^2 N_{x\varphi}}{\partial \varphi\, \partial x} = -\frac{1}{a^3}\left(\frac{\partial^2 M_\varphi}{\partial \varphi^2} + \frac{\partial^4 M_\varphi}{\partial \varphi^4}\right) \tag{d}$$

The components of displacement can also be expressed in terms of M_φ and its derivatives. We begin with the known relations [see Eqs. (253) and (254)]

$$\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} = \frac{1}{Eh}(N_x - \nu N_\varphi) \\
\gamma_{x\varphi} &= \frac{\partial u}{a\,\partial \varphi} + \frac{\partial v}{\partial x} = \frac{2(l+\nu)}{Eh} N_{x\varphi} \\
\epsilon_\varphi &= \frac{\partial v}{a\,\partial \varphi} - \frac{w}{a} = \frac{1}{Eh}(N_\varphi - \nu N_x)
\end{aligned} \tag{e}$$

[1] This approximate theory of bending of cylindrical shells was developed by U. Finsterwalder; see *Ingr.-Arch.*, vol. 4, p. 43, 1933.

From these equations we obtain

$$\frac{\partial u}{\partial x} = \frac{1}{Eh}(N_x - \nu N_\varphi)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{Eh}\left[2(1+\nu)\frac{\partial N_{x\varphi}}{\partial x} - \frac{1}{a}\left(\frac{\partial N_x}{\partial \varphi} - \nu\frac{\partial N_\varphi}{\partial \varphi}\right)\right] \tag{f}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{Eh}\left[a\left(\nu\frac{\partial^2 N_x}{\partial x^2} - \frac{\partial^2 N_\varphi}{\partial x^2}\right) + 2(1+\nu)\frac{\partial^2 N_{x\varphi}}{\partial x\,\partial\varphi} - \frac{1}{a}\left(\frac{\partial^2 N_x}{\partial \varphi^2} - \nu\frac{\partial^2 N_\varphi}{\partial \varphi^2}\right)\right]$$

Using these expressions together with Eqs. (*b*), (*c*), and (*d*) and with the expression for the bending moment

$$M_\varphi = -\frac{D}{a^2}\left(\frac{\partial v}{\partial \varphi} + \frac{\partial^2 w}{\partial \varphi^2}\right) \tag{g}$$

we finally obtain for the determination of M_φ the following differential equation of the eighth order:

$$\begin{aligned}\frac{\partial^8 M_\varphi}{\partial \varphi^8} &+ (2+\nu)a^2\frac{\partial^8 M_\varphi}{\partial x^2\,\partial\varphi^6} + 2\frac{\partial^6 M_\varphi}{\partial \varphi^6} + (1+2\nu)a^4\frac{\partial^8 M_\varphi}{\partial x^4\,\partial\varphi^4} \\ &+ 2(2+\nu)a^2\frac{\partial^6 M_\varphi}{\partial x^2\,\partial\varphi^4} + \frac{\partial^4 M_\varphi}{\partial \varphi^4} + \nu a^6\frac{\partial^8 M_\varphi}{\partial x^6\,\partial\varphi^2} + (1+\nu)^2a^4\frac{\partial^6 M_\varphi}{\partial x^4\,\partial\varphi^2} \\ &+ (2+\nu)a^2\frac{\partial^4 M_\varphi}{\partial x^2\,\partial\varphi^2} + 12(1-\nu^2)\frac{a^6}{h^2}\frac{\partial^4 M_\varphi}{\partial x^4} = 0\end{aligned} \tag{h}$$

A particular solution of this equation is afforded by the expression

$$M_\varphi = Ae^{\alpha\varphi}\sin\frac{m\pi x}{l} \tag{i}$$

Substituting it in Eq. (*h*) and using the notation

$$\frac{m\pi a}{l} = \lambda \tag{j}$$

the following algebraic equation for calculating α is obtained:

$$\begin{aligned}\alpha^8 &+ [2 - (2+\nu)\lambda^2]\alpha^6 + [(1+2\nu)\lambda^4 - 2(2+\nu)\lambda^2 + 1]\alpha^4 \\ &+ [-\nu\lambda^6 + (1+\nu)^2\lambda^4 - (2+\nu)\lambda^2]\alpha^2 + 12(1-\nu^2)\frac{a^2}{h^2}\lambda^4 = 0\end{aligned} \tag{k}$$

The eight roots of this equation can be put in the form

$$\alpha_{1,2,3,4} = \pm(\gamma_1 \pm i\beta_1) \qquad \alpha_{5,6,7,8} = \pm(\gamma_2 \pm i\beta_2) \tag{l}$$

Beginning with the edge $\varphi = 0$ and assuming that the moment M_φ rapidly diminishes as φ increases, we use only those four of the roots (*l*) which satisfy this requirement. Then combining the four corresponding solutions (*i*), we obtain

$$M_\varphi = [e^{-\gamma_1\varphi}(C_1\cos\beta_1\varphi + C_2\sin\beta_1\varphi) + e^{-\gamma_2\varphi}(C_3\cos\beta_2\varphi + C_4\sin\beta_2\varphi)]\sin\frac{m\pi x}{l} \tag{m}$$

which gives for $\varphi = 0$

$$M_\varphi = (C_1 + C_3)\sin\frac{m\pi x}{l}$$

If instead of a single term (*i*) we take the trigonometric series

$$M_\varphi = \sum A_m e^{\alpha_m \varphi} \sin \frac{m\pi x}{l} \tag{n}$$

any distribution of the bending moment M_φ along the edge $\varphi = 0$ can be obtained. Having an expression for M_φ, the resultant forces Q_φ, N_φ, and $N_{x\varphi}$ are obtained from Eqs. (*a*), (*b*), and (*c*).

If in some particular case the distributions of the moments M_φ and the resultant forces Q_φ, N_φ, and $N_{x\varphi}$ along the edge $\varphi = 0$ are given, we can represent these distributions by sine series. The values of the four coefficients in the terms containing $\sin (m\pi x/l)$ in these four series can then be used for the calculation of the four constants $C_1, \ldots, C_4$ in solution (*m*); and in this way the complete solution of the problem for the given force distribution can be obtained.

If the expressions for u, v, and w in terms of M_φ are obtained by using Eqs. (*f*), we can use the resulting expressions to solve the problem if the displacements, instead of the forces, are given along the edge $\varphi = 0$. Examples of such problems can be found in the previously mentioned paper by Finsterwalder,[1] who shows that the approximate method just described can be successfully applied in solving important structural problems.

125. The Use of a Strain and Stress Function. In the general case of bending of a cylindrical shell, for which the ratio l/a (Fig. 260) is not necessarily large, the effect of the couples M_x and M_{xy} cannot be disregarded. On the other hand, the simplified form [Eqs. (304)] of the relations between the displacements allows the introduction of a function[2] $F(x,\varphi)$ governing the state of strain and stress of the shell. Using the notation

$$c^2 = \frac{h^2}{12a^2} \qquad \xi = \frac{x}{a} \qquad \Delta = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \varphi^2} \tag{a}$$

we can rewrite Eqs. (304) in the following form, including all three components X, Y, and Z of the external loading,

$$\begin{aligned}
\frac{\partial^2 u}{\partial \xi^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial \xi\, \partial \varphi} - \nu \frac{\partial w}{\partial \xi} &= -\frac{(1-\nu^2)a^2}{Eh} X \\
\frac{1+\nu}{2}\frac{\partial^2 u}{\partial \xi\, \partial \varphi} + \frac{\partial^2 v}{\partial \varphi^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial \xi^2} - \frac{\partial w}{\partial \varphi} &= -\frac{(1-\nu^2)a^2}{Eh} Y \\
\nu\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \varphi} - w - c^2 \Delta\Delta w &= -\frac{(1-\nu^2)a^2}{Eh} Z
\end{aligned} \tag{305}$$

The set of these simultaneous equations can be reduced to a single differential equation by putting

$$\begin{aligned}
u &= \frac{\partial^3 F}{\partial \xi\, \partial \varphi^2} - \nu \frac{\partial^3 F}{\partial \xi^3} + u_0 \\
v &= -\frac{\partial^3 F}{\partial \varphi^3} - (2+\nu)\frac{\partial^3 F}{\partial \xi^2\, \partial \varphi} + v_0 \\
w &= -\Delta\Delta F + w_0
\end{aligned} \tag{306}$$

[1] *Ibid.*

[2] Due to Vlasov, *op. cit.* Almost equivalent results, without the use of a stress function, were obtained by L. H. Donnell, *NACA Rept.* 479, 1933. See also N. J. Hoff, *J. Appl. Mechanics*, vol. 21, p. 343, 1954.

where u_0, v_0, w_0 are a system of particular solutions of the nonhomogeneous equations (305). As for the strain and stress function $F(\xi,\varphi)$, it must satisfy the differential equation

$$\Delta\Delta\Delta\Delta F + \frac{1-\nu^2}{c^2}\frac{\partial^4 F}{\partial \xi^4} = 0 \tag{307}$$

which is equivalent to the group of Eqs. (305), if $X = Y = Z = 0$.* It can be shown that in this last case not only the function F but also all displacement and strain components, as well as all stress resultants of the shell, satisfy the differential equation (307).

For the elongations, the shearing strain, and the changes of the curvature of the middle surface of the shell, the expressions (300) still hold. The stress resultants may be represented either in terms of the displacements or directly through the function F. In accordance with the simplifications leading to Eqs. (304), the effect of the displacements u and v on the bending and twisting moments must be considered as negligible. Thus, with the notation

$$K = \frac{Eh}{1-\nu^2} \qquad D = \frac{Eh^3}{12(1-\nu^2)} \tag{308}$$

the following expressions are obtained:

$$\begin{aligned}
N_x &= \frac{K}{a}\left[\frac{\partial u}{\partial \xi} + \nu\left(\frac{\partial v}{\partial \varphi} - w\right)\right] = \frac{Eh}{a}\frac{\partial^4 F}{\partial \xi^2\,\partial \varphi^2} \\
N_\varphi &= \frac{K}{a}\left(\frac{\partial v}{\partial \varphi} - w + \nu\frac{\partial u}{\partial \xi}\right) = \frac{Eh}{a}\frac{\partial^4 F}{\partial \xi^4} \\
N_{x\varphi} &= \frac{K(1-\nu)}{2a}\left(\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial \xi}\right) = -\frac{Eh}{a}\frac{\partial^4 F}{\partial \xi^3\,\partial \varphi}
\end{aligned} \tag{309}$$

$$\begin{aligned}
M_x &= -\frac{D}{a^2}\left(\frac{\partial^2 w}{\partial \xi^2} + \nu\frac{\partial^2 w}{\partial \varphi^2}\right) = \frac{D}{a^2}\left(\frac{\partial^2}{\partial \xi^2} + \nu\frac{\partial^2}{\partial \varphi^2}\right)\Delta\Delta F \\
M_\varphi &= -\frac{D}{a^2}\left(\frac{\partial^2 w}{\partial \varphi^2} + \nu\frac{\partial^2 w}{\partial \xi^2}\right) = \frac{D}{a^2}\left(\frac{\partial^2}{\partial \varphi^2} + \nu\frac{\partial^2}{\partial \xi^2}\right)\Delta\Delta F \\
M_{x\varphi} &= -M_{\varphi x} = \frac{D(1-\nu)}{a^2}\frac{\partial^2 w}{\partial \xi\,\partial \varphi} = -\frac{D}{a^2}(1-\nu)\frac{\partial^2}{\partial \xi\,\partial \varphi}\Delta\Delta F
\end{aligned} \tag{310}$$

$$\begin{aligned}
Q_x &= -\frac{D}{a^3}\frac{\partial}{\partial \xi}\Delta w = \frac{D}{a^3}\frac{\partial}{\partial \xi}\Delta\Delta\Delta F \\
Q_\varphi &= -\frac{D}{a^3}\frac{\partial}{\partial \varphi}\Delta w = \frac{D}{a^3}\frac{\partial}{\partial \varphi}\Delta\Delta\Delta F
\end{aligned} \tag{311}$$

Representing the differential equation (307) in the form

$$(\Delta)^4 F + 4\gamma^4\frac{\partial^4 F}{\partial \xi^4} = 0 \tag{b}$$

where

$$\gamma = \sqrt[4]{\frac{3(1-\nu^2)a^2}{h^2}} \tag{c}$$

* Further stress functions F_x, F_y, F_z were introduced by Vlasov, *op. cit.*, to represent the particular integral of Eqs. (305) if X, Y, or Z, respectively, is not zero.

we see that Eq. (307) is also equivalent to the group of four equations

$$\Delta F_n \pm \gamma(1 \pm i)\frac{\partial F_n}{\partial \xi} = 0 \tag{d}$$

with $i = \sqrt{-1}$ and $n = 1, 2, 3, 4$. Putting, finally,

$$\begin{aligned} F_1 &= e^{-\frac{1}{2}\gamma(1+i)\xi}\Phi_1 \\ F_2 &= e^{\frac{1}{2}\gamma(1+i)\xi}\Phi_2 \\ F_3 &= e^{-\frac{1}{2}\gamma(1-i)\xi}\Phi_3 \\ F_4 &= e^{\frac{1}{2}\gamma(1-i)\xi}\Phi_4 \end{aligned} \tag{e}$$

for the four new functions Φ_n a set of four equations

$$\Delta\Phi_n + \mu_n i\Phi_n = 0 \tag{f}$$

is obtained, in which for the constant μ_n we have to assume

$$\begin{aligned} \mu_1 = \mu_2 &= -\frac{a}{2h}\sqrt{3(1-\nu^2)} \\ \mu_3 = \mu_4 &= \frac{a}{2h}\sqrt{3(1-\nu^2)} \end{aligned} \tag{g}$$

The form of each of the equations (*f*) is analogous to that of the equation of vibration of a membrane. In comparison with Eqs. (*d*), Eqs. (*f*) have the advantage of being invariant against a change of coordinates on the cylindrical surface of the shell.

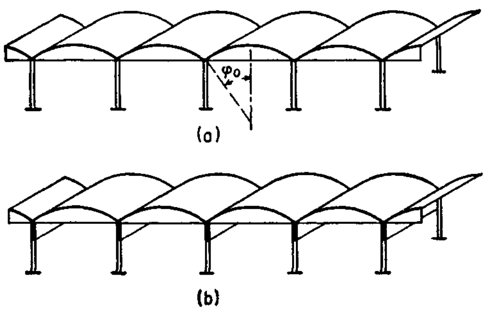

FIG. 261

126. Stress Analysis of Cylindrical Roof Shells.[1] Three typical roof layouts are shown in Figs. 261 and 265. The shells may be either continuous in the direction x or else supported only twice, say in the planes $x = 0$ and $x = l$. We shall confine ourselves to the latter case. We suppose the supporting structures to be rigid with

[1] See also "Design of Cylindrical Concrete Shell Roofs," *ASCE Manuals of Eng. Practice*, no. 31, 1952; J. E. Gibson and D. W. Cooper, "The Design of Cylindrical Shell Roofs," New York, 1954; R. S. Jenkins, "Theory and Design of Cylindrical Shell Structures," London, 1947; A. Aas-Jakobsen, "Die Berechnung der Zylinderschalen," Berlin, 1958. Many data on design of roof shells and an interesting comparison of different methods of stress analysis may be found in *Proceedings of a Symposium on Concrete Shell Roof Construction*, Cement and Concrete Association, London, 1954.

respect to forces acting in their own planes, x = constant, but as perfectly flexible with respect to transverse loading. In Fig. 261a the tension members at $\varphi = \varphi_0$ are flexible, whereas the shells shown in Figs. 261b and 265 are stiffened by beams of considerable rigidity, especially so in the vertical plane.

Any load distribution over the surface of the shell may be represented by the magnitude of its three components in the form of the series

$$
\begin{aligned}
X &= \sum_{m=1}^{\infty} X_m(\varphi) \cos \frac{\lambda_m x}{a} \\
Y &= \sum_{m=1}^{\infty} Y_m(\varphi) \sin \frac{\lambda_m x}{a} \\
Z &= \sum_{m=1}^{\infty} Z_m(\varphi) \sin \frac{\lambda_m x}{a}
\end{aligned}
\tag{a}
$$

in which

$$
\lambda_m = \frac{m\pi a}{l} \tag{b}
$$

Likewise, let us represent the particular solutions u_0, v_0, w_0 in expressions (306) in the form

$$
\begin{aligned}
u_0 &= \sum_{m=1}^{\infty} U_{om}(\varphi) \cos \frac{\lambda_m x}{a} \\
v_0 &= \sum_{m=1}^{\infty} V_{om}(\varphi) \sin \frac{\lambda_m x}{a} \\
w_0 &= \sum_{m=1}^{\infty} W_{om}(\varphi) \sin \frac{\lambda_m x}{a}
\end{aligned}
\tag{c}
$$

Expressions for the stress resultants N_x and M_x obtained from these series by means of Eqs. (309) and (310), in which $\xi = x/a$, show that the conditions (h) of Art. 123 for hinged edges are fulfilled at the supports $x = 0$ and $x = l$.

In order to obtain the general expressions for the displacements in the case

$$
X = Y = Z = 0
$$

we make use of the resolving function F (Art. 125) by taking it at first in the form

$$
F_m = e^{\alpha\varphi} \sin \frac{\lambda_m x}{a} \tag{d}
$$

Substitution of this expression in the differential equation (307) yields the following characteristic equation for α:

$$
(\alpha^2 - \lambda_m^2)^4 + \frac{1 - \nu^2}{c^2} \lambda_m^4 = 0 \tag{e}
$$

in which $c^2 = h^2/12a^2$. The eight roots of this equation can be represented in the form

$$\begin{aligned} \alpha_1 &= \gamma_1 + i\beta_1 & \alpha_5 &= -\alpha_1 \\ \alpha_2 &= \gamma_1 - i\beta_1 & \alpha_6 &= -\alpha_2 \\ \alpha_3 &= \gamma_2 + i\beta_2 & \alpha_7 &= -\alpha_3 \\ \alpha_4 &= \gamma_2 - i\beta_2 & \alpha_8 &= -\alpha_4 \end{aligned} \tag{f}$$

with real values of γ and β. Using the notation

$$\rho = \sqrt{\lambda_m}\,\sqrt[8]{\frac{1-\nu^2}{c^2}} \qquad \sigma = \frac{\lambda_m^2}{\rho^2} \tag{g}$$

we obtain

$$\begin{aligned} \gamma_1 &= \frac{\rho}{\sqrt[4]{8}}\sqrt{\sqrt{(1+\sigma\sqrt{2})^2+1}+1+\rho\sqrt{2}} \\ \gamma_2 &= \frac{\rho}{\sqrt[4]{8}}\sqrt{\sqrt{(1-\sigma\sqrt{2})^2+1}-(1-\rho\sqrt{2})} \\ \beta_1 &= \frac{1}{\gamma_1}\frac{\rho^2}{\sqrt{8}} \\ \beta_2 &= \frac{1}{\gamma_2}\frac{\rho^2}{\sqrt{8}} \end{aligned} \tag{h}$$

Returning to the series form of solution, we find that the general expression for the stress function becomes

$$F = \sum_{m=1}^{\infty} f_m(\varphi)\sin\frac{\lambda_m x}{a} \tag{i}$$

where

$$f_m(\varphi) = C_{1m}e^{\alpha_1\varphi} + C_{2m}e^{\alpha_2\varphi} + \cdots + C_{8m}e^{\alpha_8\varphi} \tag{j}$$

and $C_{1m}, C_{2m}, \ldots$ are arbitrary constants.

We are able now to calculate the respective displacements by means of the relations (306). Adding to the result the solution (c), we arrive at the following expressions for the total displacements of the middle surface of the shell:

$$\begin{aligned} u &= \sum_{m=1}^{\infty}(\lambda_m f_m'' + \nu\lambda_m^3 f_m + U_{om})\cos\frac{\lambda_m x}{a} \\ v &= \sum_{m=1}^{\infty}[(2+\nu)\lambda_m^2 f_m' - f_m''' + V_{om}]\sin\frac{\lambda_m x}{a} \\ w &= \sum_{m=1}^{\infty}(2\lambda_m^2 f_m'' - f_m'''' - \lambda_m^4 f_m + W_{om})\sin\frac{\lambda_m x}{a} \end{aligned} \tag{k}$$

where primes denote differentiation with respect to φ.

The strain and stress components now are obtained by means of expressions (300), (309), (310), and (311). In the most general case of load distribution four conditions on each edge $\varphi = \pm\varphi_0$ are necessary and sufficient to calculate the constants C_{m1}, . . . , C_{m8} associated with each integer $m = 1, 2, 3, \ldots$.

As an example, let us consider the case of a vertical load uniformly distributed over the surface of the shell. From page 460 we have

$$X = 0 \qquad Y = p \sin \varphi \qquad Z = p \cos \varphi \tag{l}$$

Hence the coefficients of the series (a) are defined by

$$\begin{aligned} X_m &= \frac{2}{l}\int_0^l X \cos \frac{\lambda_m x}{a}\,dx = 0 \\ Y_m &= \frac{2}{l}\int_0^l Y \sin \frac{\lambda_m x}{a}\,dx = \frac{4p}{m\pi}\sin\varphi \qquad (m) \\ Z_m &= \frac{2}{l}\int_0^l Z \sin \frac{\lambda_m x}{a}\,dx = \frac{4p}{m\pi}\cos\varphi \end{aligned}$$

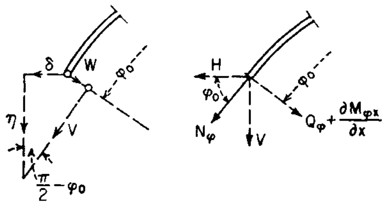

FIG. 262

in which $m = 1, 3, 5, \ldots$. An appropriate particular solution (c) is given by

$$U_{om} = A_{om} \cos \varphi \qquad V_{om} = B_{om} \sin \varphi \qquad W_{om} = C_{om} \cos \varphi \tag{n}$$

The coefficients A_{om}, B_{om}, and C_{om} are readily obtained by substitution of the expressions (c), (n), and (m) in Eqs. (305).

To satisfy the conditions of symmetry with respect to the meridian plane $\varphi = 0$, a suitable form of the function (j) is

$$\begin{aligned} f_m(\varphi) = A_{1m} \cos \beta_1\varphi \cosh \gamma_1\varphi + A_{2m} \sin \beta_1\varphi \sinh \gamma_1\varphi + A_{3m} \cos \beta_2\varphi \cosh \gamma_2\varphi \\ + A_{4m} \sin \beta_2\varphi \sinh \gamma_2\varphi \end{aligned} \tag{o}$$

in which β_1, β_2, γ_1, and γ_2 are defined by the expressions (h) and $m = 1, 3, 5, \ldots$.

In order to formulate the edge conditions on $\varphi = \pm\varphi_0$ in the simplest way, let us write the expressions for the vertical and horizontal components of the edge displacement and of the membrane forces on the edge as well (Fig. 262). We obtain

$$\eta = v \sin \varphi_0 + w \cos \varphi_0 \tag{p_1}$$
$$\delta = v \cos \varphi_0 - w \sin \varphi_0 \tag{p_2}$$
$$V = N_\varphi \sin \varphi_0 + \left(Q_\varphi + \frac{\partial M_{\varphi x}}{\partial x}\right) \cos \varphi_0 \tag{p_3}$$
$$H = N_\varphi \cos \varphi_0 - \left(Q_\varphi + \frac{\partial M_{\varphi x}}{\partial x}\right) \sin \varphi_0 \tag{p_4}$$

Finally, the rotation of the shell with respect to the edge line is expressed by

$$\chi = \frac{v}{a} + \frac{\partial w}{a\,\partial\varphi} \tag{p_5}$$

In all terms on the right-hand side of the foregoing expressions we have to put $\varphi = \varphi_0$. The following three kinds of edge conditions may be considered in particular.

Roof with Perfectly Flexible Tension Rods (Fig. 261a). Owing to many connected spans supposed to form the roof, the deformation of the roof can be considered as symmetrical with respect to the vertical plane through an intermediate edge $\varphi = \pm\varphi_0$, where the displacement δ and the rotation χ must vanish. Hence

$$v \cos \varphi_0 - w \sin \varphi_0 = 0 \tag{q_1}$$

$$v + \frac{\partial w}{\partial \varphi} = 0 \tag{q_2}$$

on $\varphi = \varphi_0$. Letting Q_0 be the weight of the tension rod per unit length, we have, by Eq. (p_3), a further condition

$$2V = Q_0 \tag{q_3}$$

in which Q_0, if constant, can be expanded in the series

$$Q_0 = \frac{4Q_0}{\pi} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m} \sin \frac{\lambda_m x}{a} \tag{p_6}$$

Finally, the elongation ϵ_x of the shell on the edge $\varphi = \varphi_0$ must be equal to the elongation of the tension member. If A_0 denotes the cross-sectional area of the latter and E_0 the corresponding Young modulus,[1] then we have, for $\varphi = \varphi_0$,

$$\frac{1}{E_0 A_0} \int_0^x 2N_{\varphi x}\, dx = \frac{\partial u}{\partial x} \tag{q_4}$$

in which the integral represents the tension force of the rod.

The further procedure is as follows. We calculate four coefficients $A_{1m}, \ldots, A_{4m}$ for each $m = 1, 3, 5, \ldots$ from the conditions (q_1), . . . , (q_4). The stress function F is now defined by Eqs. (o) and (i), and the respective displacements are given by the expressions (306) or (k). Finally, we obtain the total stress resultants by means of expressions (309) to (311), starting from the known displacements, or, for the general part of the solution, also directly from the stress function F.

Roof over Many Spans, Stiffened by Beams (Fig. 261b). The conditions of symmetry

$$v \cos \varphi_0 - w \sin \varphi_0 = 0 \tag{r_1}$$

and

$$v + \frac{\partial w}{\partial \varphi} = 0 \tag{r_2}$$

on $\varphi = \varphi_0$ are the same as in the preceding case. To establish a third condition, let Q_0 be the given weight of the beam per unit length, h_0 its depth, $E_0 I_0$ the flexural rigidity of the beam in the vertical plane, and A_0 the cross-sectional area. Then the differential equation for the deflection η of the beam becomes

$$E_0 I_0 \frac{d^4\eta}{dx^4} = Q_0 - 2V + 2\frac{h_0}{2}\frac{\partial N_{\varphi x}}{\partial x} \tag{r_3}$$

the functions η, V, and Q_0 being given by the expressions (p_1), (p_3), and (p_6), respectively. The last term in Eq. (r_3) is due to the difference of level between the edge of the shell and the axis of the beam. As for the elongation ϵ_x of the top fibers of the beam, it depends not only on the tension force but also on the curvature of the beam. Observing the effect of the curvature $d^2\eta/dx^2$, we obtain in place of Eq. (q_4) the condition

$$\frac{2}{E_0 A_0} \int_0^x N_{\varphi x}\, dx + \frac{h_0}{2}\frac{d^2\eta}{dx^2} = \frac{\partial u}{\partial x} \tag{r_4}$$

[1] In the case of a tension member composed of two materials, say steel and concrete, a transformed cross-sectional area must be used.

The further procedure of analysis remains essentially the same as in the foregoing case.

The distribution of membrane forces and bending moments M_φ obtained[1] for the middle span of a roof, comprising three such spans in all, is shown in Fig. 263. In the direction x the span of the shell is $l = 134.5$ ft, the surface load is $p = 51.8$ psf, and the weight of the beam $Q_0 = 448$ lb per ft. Stress resultants obtained by means of the membrane theory alone are represented by broken lines.

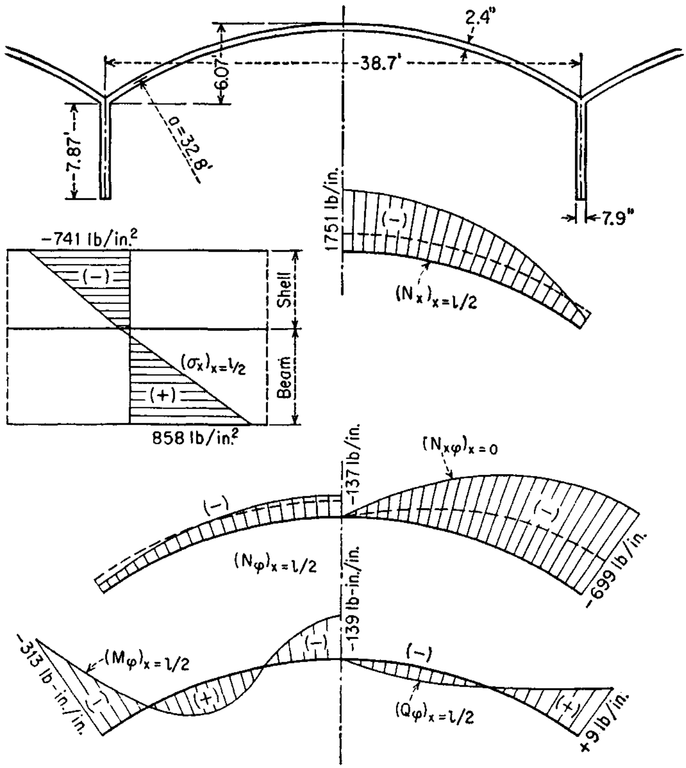

FIG. 263

One-span Roof, Stiffened by Beams (Fig. 265). In such a case we have to observe not only the deflection of the beam, given by the edge displacements η and δ, but the rotation of the beam χ as well (Fig. 264). The differential equation for the vertical deflection is, this time, of the form

$$E_0 I_0 \frac{d^4\eta}{dx^4} = Q_0 - V + \frac{h_0}{2}\frac{\partial N_{\varphi x}}{\partial x} \qquad (s_1)$$

[1] By Finsterwalder, *loc. cit.*, using the method described in Art. 124; see also *Proc. Intern. Assoc. Bridge Structural Engrs.*, vol. 1, p. 127, 1932.

the notation being the same as in the previous case. The horizontal deflection is governed in like manner by the equation

$$E_0 I'_0 \frac{d^4}{dx^4}\left(\delta - \chi \frac{h_0}{2}\right) = -H \qquad (s_2)$$

in which $E_0 I'_0$ denotes the flexural rigidity of the beam in the horizontal plane, whereas δ, χ, and H are given by the expressions (p_2), (p_5), and (p_4).

The condition of equilibrium of couples acting on an element of the beam and taken about the axis of the beam (Fig. 264) yields a further equation

$$\frac{dM_t}{dx} - \frac{Hh_0}{2} + M_\varphi = 0 \qquad (t)$$

where M_t is the torsional moment of the beam. Now, the relation between the moment M_t, the twist $\theta = \partial\chi/\partial x$, and the torsional rigidity C_0 of the beam is

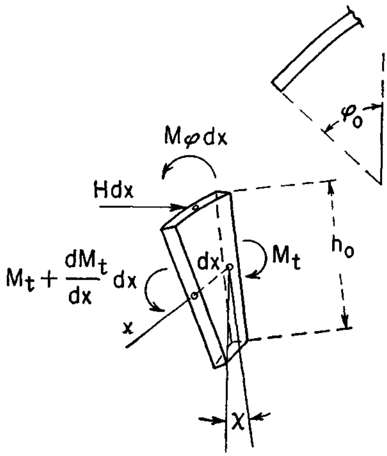

FIG. 264

$$M_t = C_0 \frac{d\chi}{dx} \qquad (u)$$

Substituting this in Eq. (t), we obtain the third edge condition

$$C_0 \frac{d^2\chi}{dx^2} - \frac{Hh_0}{2} + M_\varphi = 0 \qquad (s_3)$$

in which χ is given by the expression (p_5) and $\varphi = \varphi_0$.

The elongation ϵ_x of the top fibers of the beam due to the deflection δ may be neglected, the average value of ϵ_x through the thickness of the beam being zero. Therefore, the condition (r_4) of the foregoing case can be rewritten in the form

$$\frac{1}{E_0 A_0} \int_0^x N_{\varphi x}\, dx + \frac{h_0}{2} \frac{d^2\eta}{dx^2} = \frac{\partial u}{\partial x} \qquad (s_4)$$

Again the remaining part of the stress analysis is reduced to the determination of the constants $A_{1m}, \ldots, A_{4m}$ for each $m = 1, 3, 5, \ldots$ from Eqs. (s_1) to (s_4) and to the computation of stresses by means of the respective series.

Figure 265 shows the stress distribution in the case of a shell with $l = 98.4$ ft and $\varphi_0 = 45°$. It is seen in particular that the distribution of the membrane stresses σ_x over the depth of the whole beam, composed by the shell and both stiffeners, is far from being linear. However, by introducing $\delta = 0$ as the edge condition instead of the condition (s_2), an almost linear stress diagram 2 could be obtained. If we suppose, in addition, that the rotation χ vanishes too, we arrive at a stress distribution given by curve 3.*

* For particulars of the calculation see K. Girkmann, "Flächentragwerke," 4th ed., p. 499, Springer-Verlag, Vienna, 1956. The diagrams of Figs. 265 and 263 are reproduced from that book by permission from the author and the publisher.

Various simplifications can be introduced into the rather tedious procedure of stress calculation described above.

Thus, if the ratio l/a is sufficiently large, the stress resultants M_x, Q_x, and $M_{x\varphi}$ can be disregarded, as explained in Art. 124. Again, the particular solution (*c*) may be replaced by a solution obtained directly by use of the membrane theory of cylindrical

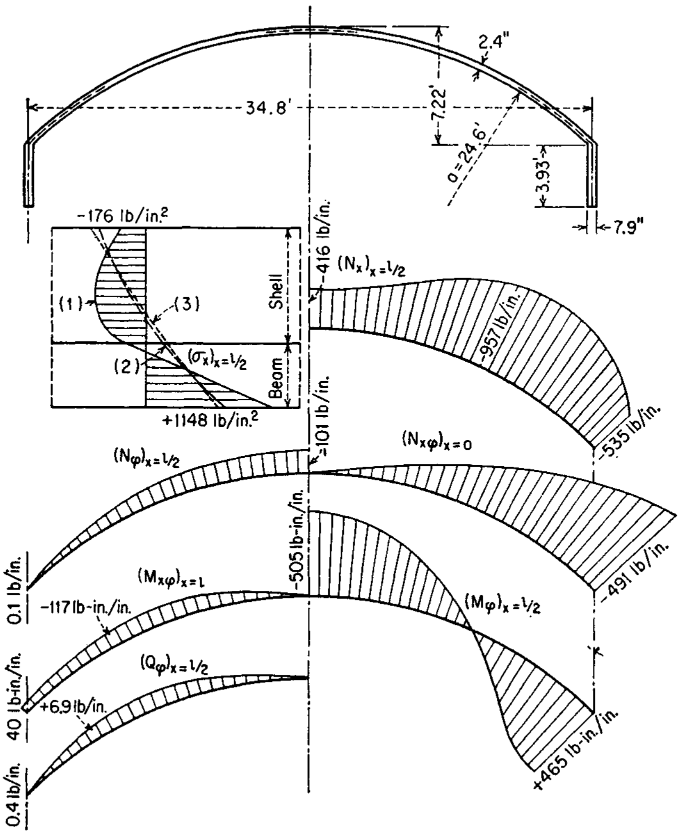

Fig. 265

shells (Art. 112). The corresponding displacements, needed for the formulation of the boundary conditions, could be obtained from Eqs. (309). The method considered in Art. 124 is simplified still more if from all derivatives with respect to φ needed to represent the strain and stress components, only those of the highest order are retained.[1]

On the other hand, the procedure of the stress computation can be greatly reduced by use of special tables for strain and stress components due to the action of the edge

[1] See H. Schorer, *Proc. ASCE*, vol. 61, p. 181, 1935.

forces on the cylindrical shell.[1] A method of iteration[2] and the method of finite differences[3] have also been used in stress analysis of shells.

If edge conditions on the supports $x = 0$, $x = l$ of the shell are other than those assumed on page 524, the stress disturbance arising from the supplementary edge forces would require special investigation.[4]

Provided l/a is not small, the roof shell may also be considered primarily as a beam.[5] Various methods of design of such a beam are based on different assumptions with respect to the distribution of membrane forces N_x over the depth of the beam. A possible procedure, for example, is to distribute the membrane forces along the contour of the shell according to the theory of elasticity and to distribute them along the generatrices according to the elementary beam theory.

In the case of very short roof shells continuous over many supports, the edge conditions on $\varphi = \pm\varphi_0$ become secondary, and a further simplification of the stress analysis proves possible.[6]

So far only circular cylindrical shells have been considered; now let us consider a cylindrical shell of any symmetrical form (Fig. 266). Given a vertical loading varying only with the angle φ, we always can obtain a cylindrical surface of pressure going through the generatrices A, C, and B. If, for instance, the load is distributed uniformly over the ground plan of the shell, the funicular curve ACB would be a parabola. Now suppose the middle surface of the shell to coincide with the surface of pressure due to a given load. The total load then is transmitted by the forces N_φ toward the edges A and B of the shell to be carried finally by the side beams over the whole length of the cylinder. If, instead, we want the load to be transmitted toward the end supports of the shell by the action of the membrane forces N_x and $N_{x\varphi}$, a shell contour overtopping the funicular (thrust-line) curve must be chosen (Fig. 266).

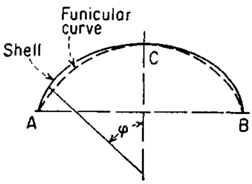

FIG. 266

From the relation $N_\varphi = -Za$ [see Eqs. (270)] we also conclude that for a vertical load, i.e., for $Z = p_v \cos\varphi$, we have $N_\varphi = -p_v a \cos\varphi$, where p_v is the intensity of the load. Therefore the ring forces N_φ on the edge vanish only when $\varphi_0 = \pi/2$, that is, when the tangents to the contour line of the shell are vertical at the edges A and B. This condition is satisfied by such contours as a semicircle, a semiellipse, or a cycloid,[7] which all overtop the pressure line due to a uniformly distributed load.

[1] Such tables (for $\nu = 0.2$) are given by H. Lundgren in his book "Cylindrical Shells," vol. 1, Copenhagen, 1949. For tables based on a simplified differential equation, due to L. H. Donnell, see D. Rüdiger and J. Urban, "Kreiszylinderschalen," Leipzig, 1955. See also references, page 524.

[2] A. Aas-Jakobsen, *Bauingenieur*, vol. 20, p. 394, 1939.

[3] H. Hencky, "Neuere Verfahren in der Festigkeitslehre," Munich, 1951. For the first application of the method to stress analysis of shells, see H. Keller, *Schweiz. Bauztg.*, p. 111, 1913. The relaxation method has been applied to stress analysis of shells by W. Flügge, "Federhofer-Girkmann-Festschrift," p. 17, Vienna, 1950.

[4] By application of Miesel's theory, *op. cit.*, or by an approximate method due to Finsterwalder, *op. cit.*

[5] This approach has especially been used by A. Aas-Jakobsen, *op. cit.*, p. 93.

[6] See B. Thürlimann, R. O. Bereuter, and B. G. Johnston, *Proc. First U.S. Natl. Congr. Appl. Mech.*, 1952, p. 347. For application of the photoelasticity method to a cylindrical shell (tunnel tube), see G. Sonntag, *Bauingenieur*, vol. 31, p. 408, 1956.

[7] For membrane stresses in shells of this kind see, for example, Girkmann, *op. cit.*, and A. Pflüger, "Elementare Schalenstatik," Berlin, 1957. The bending of semielliptical shells was considered by A. Aas-Jakobsen, *Génie civil*, p. 275, 1937. For other shapes of cylindrical roofs, see E. Wiedemann, *Ingr.-Arch.*, vol. 8, p. 301, 1937.